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# The Solution to Some Hypersingular Integral Equations

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**Abstract:** The solution to integral equations $b(t)=f(t)+\int_0^t (t-s)^{\lambda-1} b(s)ds$ is given explicitly for λ <0 for the first time. For λ <0 the kernel of the integral equation is hypersingular and the integral diverges classically. Therefore, the above equation was considered as an equation that did not make sense. The author gives a definition of the divergent integral in the above equation. The Laplace transform is used in this definition and in a study of this equation. Sufficient conditions are given for a function *F*(*p*) to be a Laplace transform of a function *f*(*t*) or of a tempered distribution *f*. These results are newl and their proofs are also novel.

**Keywords:** Integral Equations with Hypersingular Kernels, Laplace Transform

## Introduction

Consider the equation:

$$b(t)=f(t)+\int_0^t (t-s)^{\lambda-1} b(s)ds, \quad t\ge 0 \tag{1}$$

If $\lambda$ <0 then the integral in Eq. (1) diverges classically, that is, from the classical analysis point of view. There were no publications in which this class of integral equations with hypersingular integrals was discussed, except in the author's monographs (Ramm, 2021; 2023). In particular, a study of Eq. (1) with $\lambda=-\frac{1}{4}$ the author solved in the monograph (Ramm, 2023) the millennium problem related to the Navier-Stokes equations.

One of our goals is to define this integral for $\lambda$ <0. $\lambda=-\frac{1}{4}$ This integral equation was studied in Ramm (2023). Our ideas and proofs are different from those in Ramm (2023). In Ramm (2021; 2023) applications of our results for the case when $\lambda=-\frac{1}{4}$ are given.

Let:

$$L(f):=F(p):=\int_0^\infty e^{-pt} f(t)dt, \quad p=\sigma+is, \ \sigma\ge\sigma_0 \tag{2}$$

Let us define $L\left(\int_0^t (t-s)^{\lambda-1} b(s)ds\right)$ for $\lambda$ <0. The integral $\int_0^t (t-s)^{\lambda-1} b(s)ds := t^{\lambda-1}\star b$ is a convolution.

We define this integral for λ <0 by analytic continuation from the region *Re* $\lambda$ >0. Namely, we define the Laplace transform of this convolution and define the convolution integral as the inverse Laplace transform $L^{-1}L(t^{\lambda-1}\star b)$.

Let:

$$\Phi_\lambda(t):=\frac{t^{\lambda-1}}{\Gamma(\lambda)} \tag{3}$$

where, $\Gamma(\lambda)$ is the Gamma function and $t^{\lambda-1}:=t_+^{\lambda-1}$, that is, $t^{\lambda-1}=0$ for $t$ <0. One has:

$$\int_0^t (t-s)^{\lambda-1} b(s)ds = t^{\lambda-1}\star b = \Gamma(\lambda)\Phi_\lambda \star b \tag{4}$$

where the $\star$ denotes the convolution.

One has (Ramm, 2023):

$$L(\Phi_\lambda)=\frac{1}{p^\lambda}, \quad \forall\lambda\in\mathbb{C} \tag{5}$$

where ℂ is the complex plane.

The Γ(λ) is analytic for all $\lambda\in\mathbb{C}$ except for the points $\lambda=0,-1,-2,\ldots$.

To define the convolution of two distributions, we first define their direct product $f\times h$ (Gel'fand and Shilov, 1964):

$$\left(f\times h,\phi(x,y)\right)=\left(f,\left(h,\phi(x,y)\right)\right)$$

where, $\phi(x,y)$ is a test function, $(h,\phi(x,y))$ is the test function of $x$. The direct product has these properties (Gel'fand and Shilov, 1964):

$$f \times h = f \times h,$$
$$f \times \{h \times g\} = \{f \times h\} \times g$$

Let us define the convolution of two distributions by the formula (Gel'fand and Shilov, 1964):

$$\left(f \star h, \phi(x)\right) = \left(f \times h, \phi(\xi + \eta)\right) \tag{6}$$

This definition makes sense if the supports of $f$ and $h$ vanish to the left of some point. In our paper, this point is $x = 0$.

It is known (Schiff, 1999) that:

$$L(f \star h) = L(f)L(h) \tag{7}$$

Therefore:

$$L(\Phi_\lambda \star b) = \frac{1}{p^\lambda} L(b), \quad \forall \lambda \in \mathbb{C} \tag{8}$$

Taking the Laplace transform of Eq. (1) and using formula (8), one obtains the Laplace transform of the solution to Eq. (1):

$$L(b) = \frac{L(f)}{1 - \frac{\Gamma(\lambda)}{p^\lambda}} \tag{9}$$

An important question is:

For what $f = f(t)$ formula (9) is a Laplace transform of an integrable function rather than a distribution?

We answer this question in Theorem 1.

In the published works (Schiff, 1999) the assumptions are usually made on $f$, for example, $f(t) = 0$ for $t < 0$, $|f(t)| \le ce^{at}$ for some $a > 0$, $|F(p)| \le \frac{c}{p-a}$ for $\sigma > a$, $p = \sigma + is$, $\sigma, s \in \mathbb{R}$.

The novel feature of our result is the total absence of any assumptions about $f$. In particular, we do not even assume that $f(t) = 0$ for $t < 0$.

**Theorem 1.** Assume that:

a) $F(p)$ is analytic in the half-plane $\sigma > \sigma_0$,
b) $\lim_{|p|\to\infty, \sigma \ge \sigma_0} F(p) = 0$,
c) There exist for almost all $\sigma_0$ the limits $\lim_{\sigma\to\sigma_0} F(\sigma + is) = F(\sigma_0 + is)$ uniformly with respect to $s \in \mathbb{R}$ and:

$$\int_{-\infty}^{\infty} |F(\sigma_0 + is)|^\mu ds \le c, \quad \mu \ge 1$$

where, $c > 0$ does not depend on $\sigma_0$.

Then there exists a unique $f = f(t)$, $f(t) = 0$ for $t < 0$, such that:

$$F(p) = \int_0^\infty e^{-pt} f(t) dt, \quad \sigma > \sigma_0 \tag{10}$$

**Remark 1.** If $\mu \in [1,2]$ then:

$$h(t) = \int_{-\infty}^{\infty} f(\sigma_0 + is) e^{ist} ds \in L^{\frac{\mu}{\mu-1}}$$

by the Hausdorff-Young theorem (Hörmander, 1998).

If $\mu > 2$, then $h(t)$ is a tempered distribution of finite order (Hörmander, 1998).

**Theorem 2.** Assume that $L(f)$ is analytic in the half-plane $\sigma > \sigma_0 \ge 0$,

$\lim_{|p|\to\infty, \sigma\ge\sigma_0} L(f) = 0$ and $\int_{-\infty}^{\infty} |f(\sigma_0 + is)|^\mu ds \le c, \mu \in [1,2]$.

If $\sigma_0 > |\lambda|$ for a fixed $-1 < \lambda < 0$, then formula (9) gives a unique solution to Eq. (1) which is a function $b(t) = e^{\sigma_0 t} h(t)$ and $h \in L^{\frac{\mu}{\mu-1}}$.

In section 2 proofs are given.

## *Proofs*

**Proof of Theorem 1**. Define:

$$f(t) = \frac{1}{2\pi i} \int_{K\sigma_0} e^{pt} F(p) dp := e^{\sigma_0 t} h(t), \quad h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} F(\sigma_0 + is) ds : \tag{11}$$

where $K_{\sigma_0} = \{\sigma_0 - i\infty, \sigma_0 + i\infty\}$, $p = \sigma_0 + is$, $dp = ids$.

By Remark 1 and the assumptions of Theorem 1, $h(t)$ is a well-defined function for $\mu \in [1,2]$ or a distribution of finite order if $\mu > 2$.

Let us prove that $f(t) = 0$ for $t < 0$.

Consider a closed contour $\mathbb{L}_n = K_{\sigma_0,n} \cup \gamma_n$, where $\gamma_n = \{\sigma_0 + ne^{i\theta}\}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $K_{\sigma_0,n} = \{\sigma_0 - in, \sigma_0 + in\}$.

Inside $\mathbb{L}_n$ the function $e^{pt}F(p)$ is analytic. By the Cauchy theorem, one has:

$$0 = \int_{L_n} e^{pt} F(p) dp = \left(\int_{-K\sigma_0,n} + \int_{\gamma n}\right) e^{pt} F(p) dp, \tag{12}$$

where $-K_{\sigma_0,n} = \{\sigma_0 - in, \sigma_0 + in\}$.

If $t<0$, then:

$$\lim_{n\to\infty} \int_{\gamma n} e^{pt} F(p) dp = 0. \tag{13}$$

Indeed:

$$\left|\int_{\gamma n} e^{pt}F(p)dp\right| \le \max_{-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}}\left|F\left(\sigma_0+ne^{i\theta}\right)\right|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\sigma_0|t|n\cos(\theta)}nd\theta \quad (14)$$

and:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-|t|n\cos(\theta)}d\theta = 2\int_0^{\frac{\pi}{2}} e^{-|t|n\sin(\phi)}d\phi \quad (15)$$

Since sin $(\phi)\ge\frac{\pi}{2}\phi$ on the interval $[0,\pi/2]$, it follows that:

$$\int_0^{\pi/2} e^{-|t|n\sin(\phi)}d\phi \le \frac{\pi}{2|t|n}\left(1-e^{-|t|n}\right)\le c \quad (16)$$

Form formulas (14-16) and assumption (b) of Theorem 1 formula (13) follow.

From formula (12) one gets:

$$\lim_{n\to\infty}\int_{K\sigma_0,n} e^{pt}F(p)dp = \int_{K\sigma_0} e^{pt}F(p)dp = 0, \quad t<0 \quad (17)$$

Therefore,

$$f(t)=0 \quad for \quad t<0 \quad (18)$$

Let us now prove formula (10).

Note that:

$$\int_{K\sigma_0} e^{pt}F(p)dp = \int_{K\sigma} e^{pt}F(p)dp, \quad \sigma\ge\sigma_0 \quad (19)$$

Indeed, by the Cauchy theorem and assumptions a) and b) of Theorem 1 it follows that:

$$\int_{M_n} e^{pt}F(p)dp = 0 \quad (20)$$

where, $\mathbb{M}n$ is the rectangular contour consisting of two vertical lines $\{\sigma - in, \sigma + in\}$ and $\{\sigma_0 + in, \sigma_0 - in\}$ and two horizontal lines $\{\sigma + in, \sigma_0 + in\}$ and $\{\sigma_0 - in, \sigma - in\}$.

Passing to the limit $n \to \infty$ and using assumptions (a) and (b) of Theorem 1, we derive formula (19).

To prove formula (10), we use the formula $f(t)=e^{\sigma_0 t}h(t)$, where:

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}F(\sigma+is)ds \quad (21)$$

and the known Fourier inversion formula:

$$F(\sigma+is)=\int_{-\infty}^{\infty} e^{-ist}h(t)dt=\int_0^{\infty} e^{-pt}f(t)dt, \quad \sigma\ge\sigma_0 \quad (22)$$

where we took into account formula (18).

Theorem 1 is proved.

**Remark 2.** Let us give another way to prove formula (18). Let us integrate formula (11), multiplied by $e^{-qt}$, *Re q* > *Re p*, over $t$ from 0 to ∞. This yields:

$$\int_0^{\infty} e^{-qt}f(t)dt=\frac{1}{2\pi i}\int_{K\sigma_0}\frac{F(p)dp}{q-p}=\frac{1}{2\pi i}\int_{-K\sigma_0}\frac{F(p)dp}{q-p} \quad (23)$$

where: $-K_{\sigma_0}=(\sigma_0+i\infty,\sigma_0-i\infty)$.

Let us prove that:

$$F(q)=\frac{1}{2\pi i}\int_{-K\sigma_0}\frac{F(p)dp}{q-p}, \quad Re\,q>\sigma_0 \quad (24)$$

By the Cauchy formula, one has:

$$F(q)=\frac{1}{2\pi i}\int_{Ln}\frac{F(p)dp}{q-p} \quad (25)$$

As before, we prove that:

$$\lim_{n\to\infty}\int_{\gamma n}\frac{F(p)dp}{q-p}=0 \quad (26)$$

Therefore:

$$\lim_{n\to\infty}\int_{-K_{\sigma_0,n}}\frac{F(p)dp}{p-q}=\int_{-K_{\sigma_0}}\frac{F(p)dp}{p-q} \quad (27)$$

Consequently, formula (24) is proved and formula (18) is obtained by a new method.

**Proof of Theorem 2.** If $-1<\lambda<0$ is fixed and $\sigma_0$ is large enough, then $|\Gamma(\lambda)|<\sigma_0$, $|\Gamma(\lambda)|<|p|^{\lambda}$ and $\left|1-\frac{\Gamma(\lambda)}{p^{\lambda}}\right|>c>0$.

Therefore, function (9) satisfies the assumptions of Theorem 1 if $L(f)$ satisfies these assumptions with $\mu\in[1,2]$. Consequently, $b(t)$ is a function of the form $b(t)=e^{\sigma_0 t}h(t)$, where $h\in L^{\frac{\mu}{\mu-1}}$.

Theorem 2 is proved.

## Conclusion

The analytical solution is given for a class of convolution integral equations with hypersingular kernels. Sufficient conditions on a function $F(p)$ are given for this function to be a Laplace transform of a function or a tempered distribution $f(t)$. No a priori assumptions on $f(t)$ are made. This emphasizes the novelty of the result. The results are presented in two theorems.

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