

Topological Beth Model and its Application to Functionals of High Types

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Abstract: Based on the definition of Beth-Kripke model by Dragalin, we describe Beth model from the topological point of view. We show the relation of the topological definition with more traditional relational definition of Beth model that is based on forcing. We apply the topological definition to construct a Beth model for a theory of intuitionistic functionals of high types and to prove its consistency.

Keywords: Intuitionistic Functional, Creating Subject, Topological Model, Beth Model, Forcing Relation.

1. Introduction

The studies of metamathematical properties of non-classical theories are based on a variety of models such as topological models, Beth model and Kripke model. Moschovakis developed a topological model (Moschovakis, 1973) for intuitionistic second-order arithmetic and developed semantics (Moschovakis, 1987) for a theory of lawless sequences. Van Dalen (1978) constructed a Beth model for intuitionistic analysis. In his book (Dragalin, 1987) Dragalin studied a general Beth-Kripke model (*BK*-model) that combines forcing from Beth model and realizability from Kripke model. The applications of *BK*-model in (Dragalin, 1987) include different versions of intuitionistic arithmetic and analysis. Most applications of the aforementioned models are for intuitionistic sequences of natural numbers (e.g. choice sequences and lawless sequences). In (Kachapova, 2014; 2015) we created a Beth model for intuitionistic functionals of high types: 1-functionals (sequences of natural numbers), 2-functionals (sequences of 1-functionals), ..., $(n + 1)$ -functionals (sequences of n -functionals). That model was based on the relational definition of Beth model by van Dalen (1978).

In this study we describe the general concept of Beth model from the topological point of view. The topological version has a simpler definition than the relational version and is consistent with the definition of a general algebraic model for an axiomatic theory as in (Dragalin, 1987). In this study we apply the topological version of Beth model to the intuitionistic theory *SLP* of high-order functionals from (Kachapova, 2015), including lawless functionals and the "creating subject". It can be seen that the topological version of Beth model

simplifies some constructions and consistency proofs. In metamathematical proofs we use classical logic.

2. Preliminary Concepts

The introductory theory in this section follows the book by Dragalin (1987).

2.1. Logical-Mathematical Language. Axiomatic Theory

Definition 2.1

A **logical-mathematical language of first order** (or **logical-mathematical language**, or just **language** in short) is defined as a sequence $\Omega = \langle Srt, Cnst, Fn, Pr \rangle$, where

- 1) *Srt* is a non-empty set of sorts of objects and for each sort $\pi \in Srt$ there is a countable collection of variables of this sort;
- 2) *Cnst* is the set of all constants of the language;
- 3) *Fn* is the set of all functional symbols of the language;
- 4) *Pr* is the set of all predicate symbols of the language.

In the language Ω we can construct terms, atomic formulas and formulas as usual.

Definition 2.2

Axiomatic theory (or just **theory** in short) is defined as $Th = \langle \Omega, l, A \rangle$, where each of the three objects is described as follows.

- 1) Ω is a logical-mathematical language.
- 2) *l* is the logic of the theory. We will use only the intuitionistic logic *HPC* (Heyting's predicate calculus).

3) A is some set of closed formulas (i.e., formulas without parameters) of the language Ω ; it is called the set of non-logical axioms of Th . When axioms are stated as non-closed formulas, it means that they must be closed by universal quantifiers over all parameters.

The notation $Th \vdash \varphi$ (formula φ is **derivable** in the theory Th) means that φ is derivable in the logic I from a finite subset of the axiom set A .

2.2. Pseudo Boolean Algebras

Definition 2.3

A pair $\langle B, \leq \rangle$ is called a **pseudo Boolean algebra (p.B.a.)** if B is a set, \leq is a binary relation on B and they satisfy the following 9 conditions.

- 1) $a \leq a$.
- 2) $a \leq b \ \& \ b \leq c \Rightarrow a \leq c$.

For any $a, b \in B$ there exists an element $a \wedge b \in B$ such that:

- 3) $a \wedge b \leq a, a \wedge b \leq b$,
- 4) $c \leq a \ \& \ c \leq b \Rightarrow c \leq a \wedge b$.

For any $a, b \in B$ there exists an element $a \vee b \in B$ such that:

- 5) $a \leq a \vee b, b \leq a \vee b$,
- 6) $a \leq c \ \& \ b \leq c \Rightarrow a \vee b \leq c$.

For any $a, b \in B$ there exists an element $(a \supset b) \in B$ such that:

- 7) $a \wedge (a \supset b) \leq b$,
- 8) $c \wedge a \leq b \Rightarrow c \leq (a \supset b)$.
- 9) There exists an element $\perp \in B$ such that for any $a \in B: \perp \leq a$.

The element \perp is "falsity" and the element $T = (\perp \supset \perp)$ is "truth".

Definition 2.4

Suppose $\langle B, \leq \rangle$ is a p.B.a., $W \subseteq B$ and $a \in B$.

- 1) a is denoted $\bigwedge W$ and is called the **intersection** or **conjunction** of W if
 - a) $(\forall c \in W) (a \leq c)$ and
 - b) for any $d \in B, (\forall c \in W) (d \leq c) \Rightarrow d \leq a$.
- 2) a is denoted $\bigvee W$ and is called the **union** or **disjunction** of W if
 - a) $(\forall c \in W) (c \leq a)$ and
 - b) for any $d \in B, (\forall c \in W) (c \leq d) \Rightarrow a \leq d$.

Definition 2.5

A p.B.a. $\langle B, \leq \rangle$ is called **complete** if for any $W \subseteq B$ there exist $\bigwedge W \in B$ and $\bigvee W \in B$.

2.3. Algebraic Model of a Language

The following is definition of an algebraic model with constant domains. For brevity we call it just an algebraic model.

Definition 2.6

An **algebraic model** of the language Ω is a sequence $A = \langle B, D, \widehat{Cnst}, \widehat{Fn}, \widehat{Pr} \rangle$ of objects defined as follows.

- 1) B is a complete p.B.a.
- 2) To each sort π of the language Ω the function D assigns a non-empty set D_π , which is called the **domain** of elements of sort π .
- 3) To each constant of sort π the function \widehat{Cnst} assigns an element $\bar{c} \in D_\pi$.
- 4) Function \widehat{Fn} assigns values to functional symbols of Ω in the following way.

Let f be a functional symbol of sort π with arguments of sorts π_1, \dots, π_k . Then for any $q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k}, q \in D_\pi, \widehat{Fn}(f, q_1, \dots, q_k, q) \in B$.

\widehat{Fn} satisfies the following conditions (1) and (2).

$$\bigvee \{ \widehat{Fn}(f, q_1, \dots, q_k, q) \mid q \in D_\pi \} = T. \tag{1}$$

If $q \neq q'$, then:

$$\widehat{Fn}(f, q_1, \dots, q_k, q) \wedge \widehat{Fn}(f, q_1, \dots, q_k, q') = \perp. \tag{2}$$

- 5) If P is a predicate symbol of Ω with arguments of sorts π_1, \dots, π_k , then for any $q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k}$:

$$\widehat{Pr}(P, q_1, \dots, q_k) \in B. \tag{3}$$

An **evaluated term** is a term of the language Ω , in which all parameters are replaced by elements from suitable domains. An **evaluated formula** is a formula of the language Ω , in which all parameters are replaced by elements from suitable domains.

Definition 2.7

Suppose t is an evaluated term of sort π and $q \in D_\pi$. The set $\|t \sim q\|$ is defined by induction on the complexity of t .

1) If t is an element of D_π , then

$$\|t \sim q\| = \begin{cases} T & \text{if } t = q, \\ \perp & \text{otherwise.} \end{cases}$$

2) If t is a constant c and $\bar{c} = \widehat{Cnst}(c)$, then

$$\|t \sim q\| = \begin{cases} T & \text{if } \bar{c} = q, \\ \perp & \text{otherwise.} \end{cases}$$

3) Suppose t is $f(t_1, \dots, t_k)$, where each t_i is an evaluated term of sort π_i , $i = 1, \dots, k$. Then

$$\|t \sim q\| = \vee \left\{ \widehat{Fn}(f, q_1, \dots, q_k, q) \wedge \|t_1 \sim q_1\| \wedge \dots \wedge \|t_k \sim q_k\| \mid q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k} \right\}.$$

Lemma 2.8

For an evaluated term t of sort π :

$$\vee \{ \|t \sim q\| \mid q \in D_\pi \} = T.$$

Definition 2.9

For an evaluated formula φ , $\|\varphi\|$ is defined by induction on the complexity of φ .

1) Suppose φ is an atomic formula $P(t_1, \dots, t_k)$, where each t_i is an evaluated term of sort π_i , $i = 1, \dots, k$. Then

$$\|\varphi\| = \vee \left\{ \widehat{Pr}(P, q_1, \dots, q_k) \wedge \|t_1 \sim q_1\| \wedge \dots \wedge \|t_k \sim q_k\| \mid q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k} \right\}.$$

In particular, if each t_i is an element $p_i \in D_{\pi_i}$, then

$$\|\varphi\| = \widehat{Pr}(P, p_1, \dots, p_k).$$

- 2) $\|\psi \wedge \eta\| = \|\psi\| \wedge \|\eta\|.$
- 3) $\|\psi \vee \eta\| = \|\psi\| \vee \|\eta\|.$
- 4) $\|\psi \supset \eta\| = \|\psi\| \supset \|\eta\|.$
- 5) $\|\perp\| = \perp.$
- 6) $\|\forall x \psi(x)\| = \wedge \{ \|\psi(q)\| \mid q \in D_\pi \}$, where x is a variable of sort π .
- 7) $\|\exists x \psi(x)\| = \vee \{ \|\psi(q)\| \mid q \in D_\pi \}$, where x is a variable of sort π .

Theorem 2.10. Soundness Theorem.

Suppose φ is a formula of Ω and φ' is φ evaluated by elements of appropriate domains. Then the following hold.

1) $HPC \vdash \varphi \Rightarrow \|\varphi'\| = T.$

2) If φ is a closed formula, then

$$HPC \vdash \varphi \Rightarrow \|\varphi\| = T.$$

2.4. Topological Space

Definition 2.11

A pair $Y = \langle X, S \rangle$ is called a **topological space** if it satisfies the following conditions:

- 1) S is a collection of subsets of X ,
- 2) $\emptyset \in S, X \in S$,
- 3) $A, B \in S \Rightarrow A \cap B \in S$,
- 4) $W \subseteq S \Rightarrow \cup W \in S$.

Elements of S are called **open sets** and S is called the **topology** on X .

Definition 2.12

Suppose $Y = \langle X, S \rangle$ is a topological space. A collection H of subsets of X is called a **base of the topology** S if every $A \in S$ can be written as a union of elements of H .

Then we say that H **generates** the topology S .

Lemma 2.13

Suppose H is a collection of subsets of X . H is a base of some topology on X if it satisfies the following two conditions:

- 1) $X = \cup H$,
- 2) $(\forall A_1, A_2 \in H)(\forall x \in A_1 \cap A_2)(\exists A_3 \in H)(x \in A_3 \subseteq A_1 \cap A_2)$.

Lemma 2.14

Suppose $Y = \langle X, S \rangle$ is a topological space. Then $\langle S, \subseteq \rangle$ is a complete p.B.a. with the operations given by:

$$\begin{aligned} \perp &= \emptyset, \quad T = X, \\ A \wedge B &= A \cap B, \quad A \vee B = A \cup B, \\ A \supset B &= \{x \in X \mid (\exists C \in S)(x \in C \ \& \ C \cap A \subseteq B)\}, \\ \wedge W &= \cap W, \quad \vee W = \cup W. \end{aligned}$$

2.5. Completion Operator

Definition 2.15

Suppose $Y = \langle B, \leq \rangle$ is a p.B.a. and $\mathfrak{C} : B \rightarrow B$.

1) \mathfrak{C} is called a **completion operator** on Y if the following 4 conditions are satisfied for any $a, b \in B$:

- a) $a \leq \mathfrak{C} a$,
- b) $a \leq b \Rightarrow \mathfrak{C} a \leq \mathfrak{C} b$,
- c) $\mathfrak{C} \mathfrak{C} a = \mathfrak{C} a$,
- d) $a = \mathfrak{C} a \wedge a \leq \mathfrak{C} b \Rightarrow a \leq \mathfrak{C} (a \wedge b)$.

- 2) An element $a \in B$ is called **complete** if $\mathfrak{C} a = a$.
- 3) Denote $Cl(B)$ the set of all complete elements of B .

Lemma 2.16

Suppose $Y = \langle B, \leq \rangle$ is a p.B.a. and \mathfrak{C} is a completion operator on Y . Denote $Y^+ = \langle Cl(B), \leq \rangle$. Then Y^+ is also a p.B.a. with the operations given by:

$$\begin{aligned} \perp^+ &= \mathfrak{C}(\perp), \\ a \wedge^+ b &= a \wedge b, \\ a \vee^+ b &= \mathfrak{C}(a \vee b), \\ a \supset^+ b &= a \supset b. \end{aligned}$$

If Y is a complete p.B.a., then Y^+ is also a complete p.B.a. and

$$\begin{aligned} \wedge^+ W &= \wedge W, \\ \vee^+ W &= \mathfrak{C}(\vee W). \end{aligned}$$

3. Beth Model

The following definitions of Beth frame, Beth algebra and Beth model are modified from the definitions of Beth-Kripke frame, Beth-Kripke algebra and Beth-Kripke model given in the book (Dragalin, 1987). Some accompanying lemmas and theorems are proven here; other proofs can be found in (Dragalin, 1987).

3.1. Beth Frame

A **tree** is a set M with partial order \leq such that for any $x \in M$ the set $\{y \in M \mid y > x\}$ is a well-ordered set. We fix a tree $\langle M, \leq \rangle$ till the end of this section.

Definition 3.1

- 1) A **path** in M is a maximal linearly ordered subset of M .
- 2) A **path through an element** x is a path containing x .
- 3) For any $a \in M$ denote

$$\mathcal{Q}(a) = \{\text{all paths through } a\}.$$

The triple $\langle M, \leq, \mathcal{Q} \rangle$ is called a **Beth frame**.

3.2. Beth Algebra

Definition 3.2

- 1) For any $x \in M$, $[x] = \{y \in M \mid y \leq x\}$ is called the **peaked cone** generated by x

- 2) Collection of all sets $[x], x \in M$, satisfies the conditions of Lemma 2.13, therefore it is a base of a topology on M . This topology is called the **order topology**
- 3) Thus, we get a topological space $\langle M, Op(M) \rangle$, where $Op(M)$ denotes the set of all open subsets of M in the order topology

By Lemma 2.14, $\langle Op(M), \subseteq \rangle$ is a complete p.B.a.

Lemma 3.3

For any subset A of M :

$$\begin{aligned} A \text{ is open in the order topology} \\ \Leftrightarrow (\forall a \in A)(\forall b \in M)(b \leq a \Rightarrow b \in A). \end{aligned}$$

Proof

Follows from the definition. \square

Definition 3.4

For any $U \subseteq M$ denote:

$$CU = \{a \in M \mid (\forall S \in \mathcal{Q}(a))(S \cap U \neq \emptyset)\}.$$

Lemma 3.5

The operator C defined above has the following properties.

- 1) $C: Op(M) \rightarrow Op(M)$.
- 2) C is a completion operator on the p.B.a. $\langle Op(M), \subseteq \rangle$.
- 3) For any open subsets A and B of the set M :

$$CA \cap CB = C(A \cap B).$$

Definition 3.6

Denote $Y = \langle Op(M), \subseteq \rangle$. It is a complete p.B.a. The operator C from Definition 3.4 is a completion operator.

Thus, by Lemma 2.16, $Y^+ = \langle Cl(Op(M)), \subseteq \rangle$ is a complete p.B.a. It is denoted $\mathbf{B}(M, \leq, \mathcal{Q})$ and is called the **Beth algebra** generated by the Beth frame $\langle M, \leq, \mathcal{Q} \rangle$.

Theorem 3.7. Operations in Beth Algebra.

Operations in the Beth algebra are given by:

- 1) $U \wedge V = U \cap V$,
- 2) $U \vee V = C(U \cup V)$,
- 3) $U \supset V = \{a \in M \mid (\forall b \leq a)(b \in U \Rightarrow b \in V)\}$,
- 4) $T = M$,
- 5) $\perp = \{a \in M \mid \mathcal{Q}(a) = \emptyset\}$,
- 6) $\wedge W = \cap W$,
- 7) $\vee W = C(\cup W)$.

Proof

Follows from Lemma 2.16. \square

Lemma 3.8

1) For any subsets A and B of the set M :

$$CA \vee CB = C(A \cup B).$$

2) For any collection W of subsets of M :

$$\bigvee \{CA \mid A \in W\} = C(\bigcup W).$$

Proof

1) We need to prove:

$$C(CA \cup CB) = C(A \cup B).$$

By properties of completion operator, $A \subseteq CA$, $A \cup B \subseteq CA \cup CB$ and $C(A \cup B) \subseteq C(CA \cup CB)$.

To prove the inverse, it is sufficient to show that $CA \cup CB \subseteq C(A \cup B)$.

Since $A \subseteq A \cup B$, we have $CA \subseteq C(A \cup B)$. Similarly $CB \subseteq C(A \cup B)$ and $CA \cup CB \subseteq C(A \cup B)$.

2) is proven similarly. \square

3.3. Beth Model

Beth model is a particular case of an algebraic model.

We will use the notation $f: a \Rightarrow b$, which means that f is a partial function from set a to set b . The notation $Z \downarrow$ means that the object Z is defined.

Definition 3.9

A **Beth model** for a language $\Omega = \langle Srt, Cnst, Fn, Pr \rangle$ is an algebraic model $\langle B, D, \widehat{Cnst}, \widehat{Fn}, \widehat{Pr} \rangle$ with specific definitions of B , \widehat{Fn} and \widehat{Pr} .

1) $B = \mathbf{B}(M, \leq, \mathbf{Q})$ is the Beth algebra generated by the Beth frame $\langle M, \leq, \mathbf{Q} \rangle$.

Elements of M are denoted $\alpha, \beta, \gamma, \dots$.

2) Before defining \widehat{Fn} we define a function \overline{Fn} .

To each $\alpha \in M$ and each functional symbol $f(x_1, \dots, x_k)$ with sort π and arguments of sorts π_1, \dots, π_k , respectively, the function \overline{Fn} assigns a partial function $f^{[\alpha]}: D_{\pi_1} \times \dots \times D_{\pi_k} \Rightarrow D_{\pi}$ such that for any $q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k}, q \in D_{\pi}$:

$$\{\alpha \mid f^{[\alpha]}(q_1, \dots, q_k) = q\} \text{ is an open set and} \quad (4)$$

$$C(\{\alpha \mid f^{[\alpha]}(q_1, \dots, q_k) \downarrow\}) = M, \quad (5)$$

where C is from Definition 3.4.

\widehat{Fn} is defined by:

$$\widehat{Fn}(f, q_1, \dots, q_k, q) = C(\{\alpha \mid f^{[\alpha]}(q_1, \dots, q_k) = q\}).$$

Clearly, \widehat{Fn} satisfies the conditions (1) and (2) of the Definition 2.6 of an algebraic model.

3) Before defining \widehat{Pr} we define a function \overline{Pr} .

To each predicate symbol $P(x_1, \dots, x_k)$ with arguments of sorts π_1, \dots, π_k , respectively and $q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k}$ the function \overline{Pr} assigns an open set $\overline{Pr}(P, q_1, \dots, q_k)$.

\widehat{Pr} is defined by:

$$\widehat{Pr}(P, q_1, \dots, q_k) = C(\overline{Pr}(P, q_1, \dots, q_k)).$$

Clearly, \widehat{Pr} satisfies the condition (3) of the Definition 2.6 of an algebraic model.

This completes the definition of Beth model.

Notes

- 1) The component B of the Beth model is uniquely defined by the tree $\langle M, \leq \rangle$.
- 2) The component \widehat{Fn} is uniquely defined by the function \overline{Fn} .
- 3) The component \widehat{Pr} is uniquely defined by the function \overline{Pr} .
- 4) Thus, to construct a Beth model, one needs to specify a tree $\langle M, \leq \rangle$, a domain function D and functions \widehat{Cnst} , \widehat{Fn} and \widehat{Pr} .

Definition 3.10

For any α and evaluated term t of sort π we define $t^{[\alpha]}$ by induction on the complexity of t .

- 1) If t is an element of D_{π} , then $t^{[\alpha]} = t$.
- 2) If t is a constant c , then $t^{[\alpha]} = \widehat{Cnst}(c) = \bar{c}$.
- 3) If t is $f(t_1, \dots, t_k)$, then $t^{[\alpha]} = f^{[\alpha]}(t_1^{[\alpha]}, \dots, t_k^{[\alpha]})$.

Since $f^{[\alpha]}$ is a partial function, $t^{[\alpha]}$ is not always defined.

The set $\|t \sim q\|$ was defined in Definition 2.7 for a general algebraic model. Next lemma specifies it for the Beth model.

Lemma 3.11

Suppose t is an evaluated term of sort π and $q \in D_\pi$. Then

- 1) $\{\alpha \mid t^{[\alpha]} = q\}$ is an open set;
- 2) $\|t \sim q\| = C(\{\alpha \mid t^{[\alpha]} = q\})$.

Proof

1) Proof is by induction on the complexity of t .

If t is an element of D_π , then $\{\alpha \mid t^{[\alpha]} = q\}$ is either T or \perp ; in both cases it is an open set.

When t is a constant c , the proof is similar.

Suppose t is $f(t_1, \dots, t_k)$, where each t_i is an evaluated term of sort π_i , $i = 1, \dots, k$.

Then $f^{[\alpha]}(t_1^{[\alpha]}, \dots, t_k^{[\alpha]}) = q$ implies that each $t_i^{[\alpha]}$ is defined.

So

$$\{\alpha \mid t^{[\alpha]} = q\} = \bigcup_{\substack{q_1 \in D_{\pi_1}, \\ \dots \\ q_k \in D_{\pi_k}}} (\{\alpha \mid f^{[\alpha]}(q_1, \dots, q_k) = q\} \cap \{\alpha \mid t_1^{[\alpha]} = q_1\} \cap \dots \cap \{\alpha \mid t_k^{[\alpha]} = q_k\}).$$

By condition (4) of the definition of Beth model, $\{\alpha \mid f^{[\alpha]}(q_1, \dots, q_k) = q\}$ is an open set. Each $\{\alpha \mid t_i^{[\alpha]} = q_i\}$ is an open set by the inductive assumption.

Therefore $\{\alpha \mid t^{[\alpha]} = q\}$ is an open set.

2) Proof is by induction on the complexity of t .

If t is an element of D_π , then $t^{[\alpha]} = t$. So each side of the equality is $\begin{cases} T & \text{if } t = q, \\ \perp & \text{otherwise.} \end{cases}$

If t is a constant c , then $t^{[\alpha]} = \bar{c}$. So each side of the equality is $\begin{cases} T & \text{if } \bar{c} = q, \\ \perp & \text{otherwise.} \end{cases}$

Suppose t is $f(t_1, \dots, t_k)$, where each t_i is an evaluated term of sort π_i , $i = 1, \dots, k$.

Then

$$\{\alpha \mid t^{[\alpha]} = q\} = \bigcup_{\substack{q_1 \in D_{\pi_1}, \\ \dots \\ q_k \in D_{\pi_k}}} (\{\alpha \mid t_1^{[\alpha]} = q_1 \& \dots \& t_k^{[\alpha]} = q_k \& f^{[\alpha]}(q_1, \dots, q_k) = q\}). \tag{6}$$

By the inductive assumption each $\|t_i \sim q_i\| = C(\{\alpha \mid t_i^{[\alpha]} = q_i\})$. By part 1), each $\{\alpha \mid t_i^{[\alpha]} = q_i\}$ is an open set. Then by Lemma 3.5.3):

$$\begin{aligned} & \widehat{Fn}(f, q_1, \dots, q_k, q) \wedge \|t_1 \sim q_1\| \wedge \dots \wedge \|t_k \sim q_k\| \\ &= C(\{\alpha \mid f^{[\alpha]}(q_1, \dots, q_k) = q\}) \\ & \cap C(\{\alpha \mid t_1^{[\alpha]} = q_1\}) \cap \dots \cap C(\{\alpha \mid t_k^{[\alpha]} = q_k\}) \\ &= C(\{\alpha \mid f^{[\alpha]}(q_1, \dots, q_k) = q\} \\ & \cap \{\alpha \mid t_1^{[\alpha]} = q_1\} \cap \dots \cap \{\alpha \mid t_k^{[\alpha]} = q_k\}) \\ &= C(\{\alpha \mid t_1^{[\alpha]} = q_1 \& \dots \& t_k^{[\alpha]} = q_k \\ & \& f^{[\alpha]}(q_1, \dots, q_k) = q\}). \end{aligned}$$

So

$$\begin{aligned} \|t \sim q\| &= \vee \{ \widehat{Fn}(f, q_1, \dots, q_k, q) \wedge \|t_1 \sim q_1\| \wedge \dots \\ & \wedge \|t_k \sim q_k\| \mid q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k} \} \\ &= \vee \{ C(\{\alpha \mid t_1^{[\alpha]} = q_1 \& \dots \& t_k^{[\alpha]} = q_k \\ & \& f^{[\alpha]}(q_1, \dots, q_k) = q\}) \mid q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k} \} \\ &= C \left(\bigcup_{\substack{q_1 \in D_{\pi_1}, \\ \dots \\ q_k \in D_{\pi_k}}} \{\alpha \mid t_1^{[\alpha]} = q_1 \& \dots \& t_k^{[\alpha]} = q_k \& f^{[\alpha]}(q_1, \dots, q_k) = q\} \right) = C(\{\alpha \mid t^{[\alpha]} = q\}) \end{aligned}$$

by Lemma 3.8.2) and (6). \square

Lemma 3.12

For an evaluated term t of sort π :

$$C \left(\bigcup_{q \in D_\pi} \{\alpha \mid t^{[\alpha]} = q\} \right) = T.$$

Proof

By Lemma 2.8, $\vee \{\|t \sim q\| \mid q \in D_\pi\} = T$. So by Lemma 3.11.2) and Lemma 3.8.2),

$$\begin{aligned} T &= \vee \{ C(\{\alpha \mid t^{[\alpha]} = q\}) \mid q \in D_\pi \} \\ &= C \left(\bigcup_{q \in D_\pi} \{\alpha \mid t^{[\alpha]} = q\} \right). \quad \square \end{aligned}$$

Lemma 3.13

Suppose $P(t_1, \dots, t_k)$ is an atomic formula, where each t_i is an evaluated term of sort $\pi_i, i = 1, \dots, k$.

Then:

$$\|P(t_1, \dots, t_k)\| = C \left[\bigcup_{\substack{q_1 \in D_{\pi_1}, \\ \dots \\ q_k \in D_{\pi_k}}} \overline{Pr}(P, q_1, \dots, q_k) \right] \cap \{ \alpha \mid t_1^{[\alpha]} = q_1 \ \& \ \dots \ \& \ t_k^{[\alpha]} = q_k \}.$$

Proof

By Lemma 3.11.2):

$$\begin{aligned} & \widehat{Pr}(P, q_1, \dots, q_k) \wedge \|t_1 \sim q_1 \ \& \ \dots \ \& \ t_k \sim q_k\| \\ &= C(\overline{Pr}(P, q_1, \dots, q_k)) \cap C(\{ \alpha \mid t_1^{[\alpha]} = q_1 \}) \cap \dots \\ & \cap C(\{ \alpha \mid t_k^{[\alpha]} = q_k \}) \\ &= C(\overline{Pr}(P, q_1, \dots, q_k) \cap \{ \alpha \mid t_1^{[\alpha]} = q_1 \ \& \ \dots \ \& \ t_k^{[\alpha]} = q_k \}). \end{aligned}$$

So $\|P(t_1, \dots, t_k)\| = \bigvee \{ \widehat{Pr}(P, q_1, \dots, q_k) \wedge \|t_1 \sim q_1 \ \& \ \dots \ \& \ t_k \sim q_k\| \mid q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k} \}$

$$= \bigvee \left\{ C(\overline{Pr}(P, q_1, \dots, q_k) \cap \{ \alpha \mid t_1^{[\alpha]} = q_1 \ \& \ \dots \ \& \ t_k^{[\alpha]} = q_k \}) \mid q_1 \in D_{\pi_1}, \dots, q_k \in D_{\pi_k} \right\}$$

$$= C \left[\bigcup_{\substack{q_1 \in D_{\pi_1}, \\ \dots \\ q_k \in D_{\pi_k}}} (\overline{Pr}(P, q_1, \dots, q_k) \cap \{ \alpha \mid t_1^{[\alpha]} = q_1 \ \& \ \dots \ \& \ t_k^{[\alpha]} = q_k \}) \right]$$

by Lemma 3.8.2). □

Definition 3.14

A **forcing relation** \Vdash is defined by: for any α and evaluated formula φ

$$\alpha \Vdash \varphi \Leftrightarrow \alpha \in \|\varphi\|.$$

Theorem 3.15. Properties of the Forcing Relation.

- 1) Monotonicity of forcing: $\alpha \Vdash \varphi \ \& \ \beta \leq \alpha \Rightarrow \beta \Vdash \varphi$.
- 2) $\alpha \Vdash \varphi \Leftrightarrow (\forall S \in \mathbf{Q}(\alpha)) (\exists \beta \in S) (\beta \Vdash \varphi)$.
- 3) Suppose φ is an atomic formula $P(t_1, \dots, t_k)$, where each t_i is an evaluated term of sort $\pi_i, i = 1, \dots, k$. Then

$$\alpha \Vdash \varphi \Leftrightarrow (\forall S \in \mathbf{Q}(\alpha)) (\exists \beta \in S) (\exists q_1 \in D_{\pi_1}) \dots (\exists q_k \in D_{\pi_k}) [t_1^{[\beta]} = q_1 \ \& \ \dots \ \& \ t_k^{[\beta]} = q_k \ \& \ \beta \in \overline{Pr}(P, q_1, \dots, q_k)].$$

- 4) $\alpha \Vdash \varphi \wedge \psi \Leftrightarrow (\alpha \Vdash \varphi) \ \& \ (\alpha \Vdash \psi)$
- 5) $\alpha \Vdash \varphi \vee \psi \Leftrightarrow (\forall S \in \mathbf{Q}(\alpha)) (\exists \beta \in S) [(\beta \Vdash \varphi) \ \text{or} \ (\beta \Vdash \psi)]$
- 6) $\alpha \Vdash \varphi \supset \psi \Leftrightarrow (\forall \beta \leq \alpha) (\beta \Vdash \varphi \Rightarrow \beta \Vdash \psi)$
- 7) $\alpha \Vdash \perp \Leftrightarrow \perp$
- 8) $\alpha \Vdash \neg \varphi \Leftrightarrow (\forall \beta \leq \alpha) \neg (\beta \Vdash \varphi)$
- 9) $\alpha \Vdash \forall x \varphi(x) \Leftrightarrow (\forall \beta \leq \alpha) (\forall q \in D_\pi) (\beta \Vdash \varphi(q))$, where x is a variable of sort π
- 10) $\alpha \Vdash \exists x \varphi(x) \Leftrightarrow (\forall S \in \mathbf{Q}(\alpha)) (\exists \beta \in S) (\exists q \in D_\pi) (\beta \Vdash \varphi(q))$, where x is a variable of sort π

Proof

In the Beth model $\|\varphi\| \in Cl(Op(M))$, so $\|\varphi\|$ is both an open set and a complete element.

- 1) Follows from the fact that $\|\varphi\|$ is an open set
- 2) Follows from the fact that $\|\varphi\|$ is a complete element
- 3) Follows from Lemma 3.13

The rest follow from Definition 2.9 and Theorem 3.7. □

4. Axiomatic Theory of Functionals of High Types

As an application of the topological definition of Beth model, we construct a Beth model for an intuitionistic theory *SLP* of functionals of high types. We introduced this theory in (Kachapova, 2015) and constructed its Beth model using the van Dalen’s relational definition (Van Dalen, 1978). The topological definition outlined here simplifies some parts of the construction and proofs.

We define the theory *SLP* in three steps: We introduce axiomatic theories *L*, *LP* and *SLP*. In these theories variables have superscripts for types. A superscript for a variable is usually omitted when the variable is used for the second time or more in a formula (so its type is clear).

4.1. Axiomatic Theory L

The language of theory *L* has the following variables: x, y, z, \dots over natural numbers (variables of type 0) and variables of type $n (n \geq 1)$:

- F^n, G^n, H^n, \dots over n -functionals (functionals of type n);
- A^n, B^n, C^n, \dots over lawlike n -functionals;
- $\mathcal{F}^n, \mathcal{G}^n, \mathcal{H}^n, \dots$ over lawless n -functionals.

Constants: 0 of type 0 and for each $n \geq 1$ a constant K^n (an analog of 0 for type n).

Functional symbols: N^n , Ap^n ($n \geq 1$), S for successor, and $+$.

Predicate symbols: $=_n$ for each $n \geq 0$.

Terms and n -functionals are defined recursively as follows.

1. Every numerical variable is a term
2. Constant 0 is a term
3. Every variable of type n is an n -functional
4. Constant K^n is an n -functional
5. If t is a term, then St is a term
6. If t_1 and t_2 are terms, then t_1+t_2 and $t_1 \cdot t_2$ are terms
7. If Z is an n -functional, then $N^n(Z)$ is an n -functional (a successor of Z)
8. If Z is a 1-functional, t is a term, then $Ap^1(Z, t)$ is a term
9. If Z is an $(n+1)$ -functional, t is a term, then $Ap^{n+1}(Z, t)$ is an n -functional

$Ap^n(Z, t)$ is interpreted as the result of application of functional Z to term t . We also denote $Ap^n(Z, t)$ as $Z(t)$.

Here 1-functional is interpreted as a function from natural numbers to natural numbers and $(n+1)$ -functional is interpreted as a function from natural numbers to n -functionals.

Atomic formulas:

$t =_0 \tau$, where t and τ are terms;

$Z =_n V$, where Z and V are n -functionals ($n \geq 1$).

Formulas are constructed from atomic formulas using logical connectives and quantifiers. For a formula φ its sort $sort(\varphi)$ is the maximal type of parameters in φ ; it is 0 if φ has no parameters.

The theory L has intuitionistic predicate logic HPC with equality axioms and the following non-logical axioms.

1. $Sx \neq 0, Sx = Sy \supset x = y$.
2. $x + 0 = x, x + Sy = S(x + y)$.
3. $x \cdot 0 = 0, x \cdot Sy = x \cdot y + x$.
4. Induction for natural numbers:

$$\varphi(0) \wedge \forall x (\varphi(x) \supset \varphi(Sx)) \supset \forall x \varphi(x),$$

where φ is any formula of L .

The axioms 1 - 4 define arithmetic at the bottom level.

5. $K^{n+1}(x) = K^n, \neg(N^n(F^n) = K^n)$
6. $N^{n+1}(F^{n+1})(x) = N^n(F(x)), N^n(F^n) = N^n(G^n) \supset F^n = G^n$

The axioms 5 and 6 describe K^n and N^n as analogs of zero and successor function, respectively, on level n .

7. Principle of primitive recursive completeness of lawlike functions:

$$\exists A^i \forall x (A(x) = t),$$

where t is any term containing only variables of type 0 and variables over lawlike 1-functionals.

Denote L_s the fragment of the theory L with types not greater than s . The language of L_1 has one type of functionals and is essentially the language of the intuitionistic analysis FIM .

4.2. Axiomatic Theory LP

This theory is obtained from L by adding predicate symbols and axioms for the "creating subject".

Gödel numbering of symbols and expressions can be defined for the language of L . For an expression q we denote $\ulcorner q \urcorner$ the Gödel number of q in this numbering.

The language of theory LP is the language of L with an extra predicate symbol $Pv_{\ulcorner \bar{x} \urcorner, \varphi} (z, \bar{X})$ for every formula φ of L , which has all its parameters in the list \bar{X} ; here \bar{X} is a list of variables X_1, \dots, X_k . Traditionally this symbol is denoted $\vdash_z \varphi(\bar{X})$; it means that the formula $\varphi(\bar{X})$ has been proven by the "creating subject" at time z .

Axioms of the theory LP are all axioms of L , where the axiom schemata are taken for the formulas of the new language, and the following three axioms; in all of them φ is an arbitrary formula of L .

Axioms for the "creating subject":

$$(CS1) \quad (\vdash_z \varphi) \vee \neg(\vdash_z \varphi);$$

$$(CS2) \quad (\vdash_z \varphi) \supset (\vdash_{z+y} \varphi);$$

$$(CS3) \quad \exists z (\vdash_z \varphi) \equiv \varphi.$$

The language of LP_s is the language of L_s with extra predicate symbols $Pv_{\ulcorner \bar{x} \urcorner, \varphi} (z, \bar{X})$, where both φ and \bar{X} belong to the language of L_s . Axioms of the theory LP_s are all axioms of LP , which are formulas of the language LP_s .

If (A) is a formula of LP , we denote (A_s) the version of (A) with types not greater than s .

4.3. Axiomatic Theory SLP

We can introduce finite sequences in LP and for any k -functional F use the notation $\bar{F}(n) = \langle F(0), \dots, F(n-1) \rangle$. We consider the following three axioms for lawless functionals.

(LL1) The axiom of existence of lawless functionals:

$$\forall x \forall F^n \exists \mathcal{F}^n (\forall y \leq x) (\mathcal{F}(y) = F(y)).$$

(LL2) $\mathcal{F}^n = \mathcal{G}^n \vee \mathcal{F}^n \neq \mathcal{G}^n$.

(LL3) The principle of open data:

$$\varphi(\mathcal{H}^n) \supset \exists x \forall \mathcal{G}^n [\bar{\mathcal{G}}(x) = \bar{\mathcal{H}}(x) \supset \varphi(\mathcal{G})],$$

where φ is a formula of LP, $sort(\varphi) \leq n$ and φ does not have non-lawlike parameters of type n other than \mathcal{H}^n .

Denote (LL) a conjunction of the closures of (LL1), (LL2) and (LL3).

For an m -functional F of language LP and $n \leq m$ we denote $F(\underbrace{0 \dots 0}_n)$ as $F(0)^n$ for brevity. We consider the

following two **choice axioms**.

(C1) The axiom of choice for numbers:

$$\forall x \exists y \varphi(x, y) \supset \exists F^m \forall x \varphi(x, F(x)(0)^{m-1}),$$

where φ is a formula of LP and $m \geq \max(sort(\varphi), 1)$.

(C2) The axiom of choice with uniqueness:

$$\forall x \exists! G^n(x, G) \supset \exists F^m \forall x \varphi(x, F(x)(0)^{m-n-1}),$$

where φ is a formula of LP and $m \geq \max(sort(\varphi), n+1)$.

Denote (C) a conjunction of the closures of (C1) and (C2).

Bar induction axiom:

$$(BI) \quad \forall F^1 \exists x \varphi(\bar{F}(x)) \wedge \forall x \forall y [\varphi(y) \supset \varphi(y * \langle x \rangle)] \\ \wedge \forall y [\forall x \psi(y * \langle x \rangle) \supset \psi(x)] \wedge \forall y [\varphi(y) \supset \psi(y)] \supset \psi(0).$$

Axiomatic theory SLP is defined by:

$$SLP = LP + (LL) + (C) + (BI).$$

In the name of the theory SLP, S stands for strong, L for lawless and P for proof (relating to the "creating subject").

The fragment SLP_s is defined by:

$$SLP_s = LP_s + (LL_s) + (C_s) + (BI_s).$$

5. Application of the Topological Beth Model

As an application, we construct a Beth model for any fragment SLP_s ($s \geq 1$) of the theory SLP. This is sufficient

for the proof of consistency of SLP, since any formal proof in the theory SLP is finite and therefore it is a proof in some fragment SLP_s.

5.1. Notations

A sequence of elements x_0, x_1, \dots, x_n is denoted

$x = \langle x_0, x_1, \dots, x_n \rangle$ and we denote $lth(x) = n + 1$.

The symbol $*$ denotes the concatenation function:

$$\langle x_0, x_1, \dots, x_n \rangle * \langle y_0, y_1, \dots, y_k \rangle \\ = \langle x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_k \rangle.$$

For a sequence x , $\langle x \rangle_n$ denotes its n^{th} element and $\bar{x}(n)$ denotes the initial segment of x of length n for any $n < lth(x)$. So if $x = \langle x_0, x_1, \dots, x_{m-1} \rangle$ and $n \leq m$, then $\bar{x}(n) = \langle x_0, x_1, \dots, x_{n-1} \rangle$.

For a function f on natural numbers, $\bar{f}(n)$ denotes the initial segment of f of length n , that is the sequence $\langle f(0), f(1), \dots, f(n-1) \rangle$.

For a function f of two variables denote

$$f^{(x)} = \lambda y. f(x, y).$$

Next we introduce a few notations for a fixed set b .

1. $b^{(n)}$ is the set of all sequences of elements of b that have length n .
2. b^* is the set of all finite sequences of elements of b .
3. On the set b^* a partial order \leq is defined by the following:

$$y \leq x \text{ if } x \text{ is an initial segment of } y.$$

With this order b^* is a tree growing down; its root is the empty sequence $\langle \rangle$.

Suppose $\langle d, \leq \rangle$ is a partially ordered set and $f: (d \times \omega) \rightrightarrows c$.

1. f is called *monotonic* on d if for any $x, y \in d$, $(y \leq x \Rightarrow f^{(x)} \subseteq f^{(y)})$.
2. f is called *complete* on d if for any path S in d , $\bigcup \{f^{(x)} \mid x \in S\}$ is a total function on ω .

5.2. Beth Model \mathcal{B}_s for the Language L_s

Fix $s \geq 1$. To construct a Beth model \mathcal{B}_s for the language L_s , we will specify a tree $\langle M, \leq \rangle$, a domain function D and functions \widehat{Cnst} , \overline{Fn} and \overline{Pr} .

- 1) First we introduce a triple of objects $\langle a_k, d_k, \leq_k \rangle$ by induction on k ; here \leq_k is a partial order on d_k .

- $a_0 = \omega$, the set of all natural numbers
- $d_0 = \{\langle x \rangle \mid x \in a_0^*\}$; thus, each element of d_0 is a sequence of length 1
- \preceq_0 is generated by the order \leq on a_0^*

$\langle x \rangle \preceq_0 \langle y \rangle$ if $x \leq y$, that is the sequence y is an initial segment of the sequence x .

For $\langle x \rangle \in d_0$ we denote $lh(\langle x \rangle) = lth(x)$ and call it the length of $\langle x \rangle$.

For $k \geq 1$:

- $a_k = \{f \mid f : (d_{k-1} \times \omega) \Rightarrow a_{k-1} \text{ and } f \text{ is complete and monotonic}\}$;
- $d_k = \bigcup_{m=0}^{\infty} (a_0^{(m)} \times a_1^{(m)} \times \dots \times a_k^{(m)})$;
- \preceq_k is defined by:

$x \preceq_k y$ if for each $i \leq k$, $\langle x \rangle_i \leq \langle y \rangle_i$, that is the sequence $\langle y \rangle_i$ is an initial segment of the sequence $\langle x \rangle_i$.

If $x \in a_0^{(m)} \times \dots \times a_k^{(m)}$, we denote $lh(x) = m$ and call it the length of x .

We take $M = d_{s-1}$ with partial order \preceq_{s-1} , which we denote just \preceq .

With this order M is a subtree of the direct product of trees a_0^*, \dots, a_{s-1}^* . M is a non-countable tree growing down. Its root is $\varepsilon = \langle \langle \rangle, \dots, \langle \rangle \rangle$.

- Next we define a domain for each sort of variables.
 - Domain for natural numbers is $a_0 = \omega$.
 - Domain for k -functionals is a_k ($k = 1, 2, \dots, s$).
 - Domain for lawlike k -functionals is

$$b_k = \left\{ f \in a_k \mid f^{(\langle \cdot \rangle)} \text{ is a total function} \right\}.$$

- Before defining a domain for lawless functionals we introduce an auxiliary set c_k ($k \geq 1$):

$$c_k = \left\{ \xi \mid \xi : (\omega \times a_{k-1}) \rightarrow a_{k-1} \text{ and } \forall n (\xi^{(n)} \text{ is a bijection on } a_k) \right\}.$$

Elements of c_k are called k -permutations. For any k -permutation ξ we define $v_k(\xi)$ as the function $f \in a_k$ such that for any $n \in \omega$, $x \in d_{k-1}$:

$$f(x, n) = \begin{cases} \xi^{(n)}(\langle \langle x \rangle_{k-1} \rangle_n) & \text{if } n < lh(x), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Domain for lawless k -functionals is:

$$I_k = \{v_k(\xi) \mid \xi \in c_k\}, k = 1, 2, \dots, s.$$

Clearly, $b_k \subseteq a_k$ and $I_k \subseteq a_k$, $k = 1, \dots, s$. For any α and $k < s$, $lh(\alpha) = lth(\langle \alpha \rangle_k)$.

- Next we define \overline{Cnst} , that is interpretation \overline{C} for each constant C .

- $\overline{0} = 0$.
- The interpretation of constant K^k ($k = 1, \dots, s$) is given by the following:

$$\begin{aligned} \overline{K}^1(x, n) &= 0 \text{ for any } n \in \omega \text{ and } x \in d_0; \\ \overline{K}^{k+1}(x, n) &= \overline{K}^k \text{ for any } n \in \omega \text{ and } x \in d_k. \end{aligned}$$

- Next we define \overline{Fn} . This means defining a partial function $h^{[a]}$ for each functional symbol h and $a \in M$.

- If θ is \cdot or $+$, then $\theta^{[a]} = \theta$.
- A successor function S^k of type k ($k = 0, 1, \dots, s$) is defined by:

$$S^0(x) = x + 1; \quad S^{n+1}(f) = S^n \circ f.$$

Functional symbol S is interpreted by: $S^{[a]} = S^0$.
 Functional symbol N^k is interpreted by: $N^{k[a]} = S^k$.

- Functional symbol Ap^k is interpreted by:

$$Ap^{k[a]}(f, n) = f(\overline{\alpha}(k), n) \text{ for any } n \in \omega, f \in a_k, k = 1, \dots, s.$$

Thus, at each node α the values of a k -functional f depend not on the entire α but only on its first k components.

Due to the definition of a_k , the conditions (4) and (5) of the Definition 3.9 of Beth model are satisfied.

- The only predicate symbols in the language L are equalities $=_k$ ($k = 0, 1, 2, \dots$).

\overline{Pr} is defined by:

$$\overline{Pr}(p =_k q) = \begin{cases} T & \text{if } p = q, \\ \perp & \text{otherwise} \end{cases}$$

For any $p, q \in a_k$. Clearly, $\overline{Pr}(p =_k q)$ is an open set.

This completes the definition of the model \mathcal{B}_s for the language of L_s .

For the general case of Beth model we already defined:

- $Z^{[a]}$, where Z an evaluated term or n -functional (Definition 3.10) and
- the forcing relation $\alpha \Vdash \varphi$ (Definition 3.14)

5.3. Extending the Model \mathcal{B}_s to the Language SLP_s

First we extend the model to the language LP_s . For that we need to interpret the extra predicate symbols $Pv_{\bar{x}, \varphi, a}(z, \bar{X})$, where both φ and X belong to the language of L_s .

Definition 5.1

Consider a formula φ of L_s with all its parameters in the list X_1, \dots, X_k , which is denoted \bar{X} in short. For brevity we denote the predicate $Pv_{\bar{x}, \varphi, a}(z, \bar{X})$ as $P_0(z, \bar{X})$. We define:

$$\overline{Pr}(P_0, n, \bar{q}) = \{ \alpha \mid \exists \gamma [\alpha \preceq \gamma \ \& \ lh(\gamma) = n \ \& \ \gamma \Vdash \varphi(\bar{q})] \},$$

where, \bar{q} is the list q_1, \dots, q_k .

Let us show that $\overline{Pr}(P_0, n, \bar{q})$ is an open set in the order topology. Suppose $\alpha \in \overline{Pr}(P_0, n, \bar{q})$. Then there is γ such that $\alpha \preceq \gamma \ \& \ lh(\gamma) = n \ \& \ \gamma \Vdash \varphi(\bar{q})$. Then for any $\beta \preceq \alpha$ we have $\beta \preceq \gamma$, so $\beta \in \overline{Pr}(P_0, n, \bar{q})$.

The language of SLP_s is the same as the language LP_s , so the extended model \mathcal{B}_s is also a Beth model for the language of SLP_s .

Lemma 5.2

$$\alpha \Vdash P_0(n, \bar{q}) \Leftrightarrow (\forall S \in \mathcal{Q}(\alpha)) (\exists \gamma \in S) [lh(\gamma) = n \ \& \ \gamma \Vdash \varphi(\bar{q})].$$

Proof

Proof follows from the definitions. \square

5.4. Soundness of the Model \mathcal{B}_s for the Theory SLP_s

Theorem 5.3. Soundness Theorem.

$$SLP_s \vdash \varphi \Rightarrow \mathcal{B}_s \Vdash \bar{\varphi}.$$

Proof

Proof is by induction on the length of derivation of φ .

Here we provide proofs only for the case of the axioms (CS1) – (CS3) for the "creating subject". We prove these in all detail to illustrate our application of the topological Beth model. Other, more technical proofs can be found in (Kachapova, 2014; 2015).

We use the notations of Definition 5.1.

$$\text{Proof of } \mathcal{B}_s \Vdash (\overline{CS1}_s).$$

(CS1) is $(\vdash_z \varphi) \vee \neg(\vdash_z \varphi)$. It is sufficient to prove:

$$\varepsilon \Vdash P_0(n, \bar{q}) \vee \neg P_0(n, \bar{q}).$$

By Theorem 3.15.5) it is equivalent to:

$$(\forall S \in \mathcal{Q}(\varepsilon)) (\exists \alpha \in S) [\alpha \Vdash P_0(n, \bar{q}) \text{ or } \alpha \Vdash \neg P_0(n, \bar{q})].$$

Consider a path $S \in \mathcal{Q}(\varepsilon)$. Fix $\alpha \in S$ with $lh(\alpha) = n$. There are two cases: $\alpha \Vdash \varphi(\bar{q})$ or $\alpha \not\Vdash \varphi(\bar{q})$ (we use classical logic in metamathematics).

Case 1. $\alpha \Vdash \varphi(\bar{q})$.

Since $lh(\alpha) = n$, by Lemma 5.2 we have $\alpha \Vdash P_0(n, \bar{q})$.

Case 2. $\alpha \not\Vdash \varphi(\bar{q})$.

We will show that in this case $\alpha \Vdash \neg P_0(n, \bar{q})$. By Theorem 3.15.8), it is equivalent to:

$$(\forall \beta \preceq \alpha) \neg (\beta \Vdash P_0(n, \bar{q})).$$

Suppose $\beta \Vdash P_0(n, \bar{q})$ for some $\beta \preceq \alpha$. Consider an arbitrary $S_1 \in \mathcal{Q}(\beta)$. By Lemma 5.2, there is $\gamma \in S_1$ such that $lh(\gamma) = n \ \& \ \gamma \Vdash \varphi(\bar{q})$.

Since $S_1 \in \mathcal{Q}(\beta) \ \& \ \beta \preceq \alpha$, we have $\alpha \in S_1$. Both $\alpha, \gamma \in S_1$ and $lh(\alpha) = n = lh(\gamma)$, so $\alpha = \gamma$ and $\alpha \Vdash \varphi(\bar{q})$.

Contradiction. Therefore $\neg (\beta \Vdash P_0(n, \bar{q}))$:

$$\text{Proof of } \mathcal{B}_s \Vdash (\overline{CS2}_s).$$

(CS2) is $(\vdash_z \varphi) \supset (\vdash_{z+y} \varphi)$. It is sufficient to prove:

$$\varepsilon \Vdash P_0(n, \bar{q}) \supset P_0(n+m, \bar{q}).$$

Fix α . Suppose $\alpha \Vdash P_0(n, \bar{q})$. Consider $S \in \mathcal{Q}(\alpha)$. By Lemma 5.2, there is $\gamma \in S$ such that $lh(\gamma) = n$ and $\gamma \Vdash \varphi(\bar{q})$.

There is $\beta \in S$ with $lh(\beta) = n+m$. Since $\beta \preceq \gamma$, we have $\beta \Vdash \varphi(\bar{q})$ by monotonicity of forcing (Theorem 3.15.1)). This proves $\alpha \Vdash P_0(n+m, \bar{q})$.

Proof of $\mathcal{B}_s \Vdash (\overline{CS3}_s)$.

$(CS3)$ is $\exists z (\vdash_z \varphi) \equiv \varphi$.

It is sufficient to prove: $\varepsilon \Vdash \exists z P_0(z, \bar{q}) \equiv \varphi(\bar{q})$, which is equivalent to

$$\varepsilon \Vdash \exists z P_0(z, \bar{q}) \supset \varphi(\bar{q}) \tag{7}$$

and

$$\varepsilon \Vdash \varphi(\bar{q}) \supset \exists z P_0(z, \bar{q}). \tag{8}$$

Proof of (7)

Suppose for some α ,

$$\alpha \Vdash \exists z P_0(z, \bar{q}). \tag{9}$$

We need to prove $\alpha \Vdash \varphi(\bar{q})$. Consider a path $S \in \mathcal{Q}(\alpha)$. By (9), there is $\beta \in S$ and n such that $\beta \Vdash P_0(n, \bar{q})$. Then $S \in \mathcal{Q}(\beta)$ and by Lemma 5.2, there is $\gamma \in S$ with $\gamma \Vdash \varphi(\bar{q})$. Thus,

$$(\forall S \in \mathcal{Q}(\alpha)) (\exists \gamma \in S) (\gamma \Vdash \varphi(\bar{q})).$$

By Theorem 3.15.2, $\alpha \Vdash \varphi(\bar{q})$.

Proof of (8).

Suppose for some α , $\alpha \Vdash \varphi(\bar{q})$. Take $n = lh(\alpha)$. Suppose $S \in \mathcal{Q}(\alpha)$. For any $S_1 \in \mathcal{Q}(\alpha)$ there is $\gamma = \alpha$ with $lh(\gamma) = n$ & $\gamma \Vdash \varphi(\bar{q})$; so by Lemma 5.2, $\alpha \Vdash P_0(n, \bar{q})$. Hence $\alpha \Vdash \exists z P_0(z, \bar{q})$. \square

6. Conclusion

In this study we describe detailed steps of construction of Beth model from the topological point of view. The model is applied to intuitionistic functionals of high types and the theory *SLP* that includes lawless functionals, “the creating subject”, bar induction and some choice axioms; the model is used to prove the consistency of *SLP*. The topological version of Beth model and our model, in particular, can potentially be useful to investigate other intuitionistic principles with respect to functionals of high types.

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Ethics

This is a mathematical article; no ethical issues can arise after its publication.

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