

Fuzzy Bernstein-Stancu Operator of Rough I -Core of Triple Sequences

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Abstract: We are going to present and investigate some fundamental characteristics of fuzzy Bernstein-Stancu polynomials of rough I -convergent of triple sequence spaces and at the same time examine the set of all fuzzy Bernstein Stancu polynomials of rough I -limits of a triple sequence spaces and also relation between analyticness and fuzzy Bernstein Stancu polynomials of rough I -core of a triple sequence spaces.

Keywords: Ideal, Triple Sequences, Rough Convergence, Closed and Convex, Cluster Points and Rough Limit Points, Bernstein Stancu Operator, Fuzzy Numbers, Fuzzy Sets

Introduction

The idea of rough convergence was first introduced by Phu (2001; 2002; 2003) in dimensional normed spaces which is finite. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence kinds and the dependence of LIM_x^r on the roughness of degree denoted by r .

Aytar (2008a) studied of rough statistical convergence and described the rough statistical set's limit points of a given sequence and obtained two statistical convergence criteria associated with this set and prove that the set is convex and closed. Also, Aytar (2008b) studied that the fact that r -limit set of the sequence is equal to intersection of these sets and that r -core of the sequence is equal to the union of these sets. Dundar and Cakan (2014) have studied of rough ideal convergence and described the rough ideal limit set's points of a sequence. The notion of I -convergence for a triple sequence spaces which is based on the very structure of the ideal I of subsets of \mathbb{N}^3 , where \mathbb{N} is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we are going to examine some basic properties of rough I -convergence of a triple sequence spaces of fuzzy in three dimensional matrix spaces which are not earlier. We study the set of all rough I -limits of a triple sequence spaces of fuzzy and also the

relation between analyticness and rough I -core of a triple sequence spaces of fuzzy. Let us assume that K be a subset of the set of positive integers \mathbb{N}^3 and let us denote the set $K_{ij\ell} = \{(m, n, k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by:

$$\delta_3(K) = \lim_{i,j,\ell \rightarrow \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where, $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

First of all, we have applied the concept of (p, q) -calculus in approximate theory and introduced the (p, q) -analogue of Bernstein operators. Next, based on (p, q) -integers, some approximate results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p, q) -Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

Khan and Lobiyal (2015) gave a very nice application in computer-aided geometric design and used these Bernstein basis for construction of (p, q) -Bezier curves and surfaces based on (p, q) -integers which is further generalization of q -Bezier curves and surfaces.

Being motivated with the above described study on (p, q) -approximation and its application, in the present paper we are going to investigate statistical approximate characteristics of Bernstein-Stancu operators based on (p, q) -integers.

Now we recall some basic definitions about (p, q) -integers. For any given $u, v, w \in \mathbb{N}$, the (p, q) -integer $[uvw]_{p,q}$ is defined by:

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p - q} \text{ if } u, v, w \geq 1,$$

where, $0 < q < p \leq 1$. Here the (p, q) -factorial is defined by:

$$[0]_{p,q}! := 1 \text{ and } [uvw]_{p,q}! = [1]_{p,q} [2]_{p,q} \cdots [uvw]_{p,q} \text{ if } u, v, w \geq 1.$$

Also the (p, q) -binomial coefficient is defined by:

$$\binom{u}{m} \binom{v}{n} \binom{w}{k} \Big|_{p,q} = \frac{[u]_{p,q}! [v]_{p,q}! [w]_{p,q}!}{[m]_{p,q}! [u-m]_{p,q}! [n]_{p,q}! [v-n]_{p,q}! [k]_{p,q}! [w-k]_{p,q}!}$$

for every $u, v, w, m, n, k \in \mathbb{N}$ having $u \geq m, v \geq n, w \geq k$.

The formula for (p, q) -binomial expansion is as follows:

$$\begin{aligned} (ax + by)^{uvw} &= \sum_{m=0}^u \sum_{n=0}^v \sum_{k=0}^w p^{\frac{(u-m)(u-m-1)+\dots+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} \\ &\binom{u}{m} \binom{v}{n} \binom{w}{k} \Big|_{p,q} a^{(u-m)+(v-n)+(w-k)} b^{m+n+k} x^{(u-m)+(v-n)+(w-k)} y^{m+n+k}, \\ (x + y)^{uvw} &= (x + y)(px + qy)(p^2x + q^2y) \cdots \\ &(p^{(u-1)+(v-1)+(w-1)}x + q^{(u-1)+(v-1)+(w-1)}y), \\ (1 - x)^{uvw} &= (1 - x)(p - qx)(p^2 - q^2x) \cdots \\ &(p^{(u-1)+(v-1)+(w-1)} - q^{(u-1)+(v-1)+(w-1)}x), \end{aligned}$$

and:

$$(x)_{p,q}^{mnk} = x(px)(p^2x) \cdots (p^{(u-1)+(v-1)+(w-1)}x) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator having order rst is given by:

$$B_{rst}(f, x) = \sum_{m=0}^r \sum_{n=0}^s \sum_{k=0}^t f\left(\frac{mnk}{rst}\right) \binom{r}{m} \binom{s}{n} \binom{t}{k} \Big|_{p,q} x^{m+n+k} (1-x)^{(m-r)+(n-s)+(t-k)}$$

where f is a given continuous function (which may be real or complex valued one) function defined on the interval $[0, 1]$.

(p, q) -Bernstein operators are defined as follows:

$$\begin{aligned} B_{rst,p,q}(f, x) &= \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}} \sum_{m=0}^r \sum_{n=0}^s \sum_{k=0}^t \binom{r}{m} \binom{s}{n} \binom{t}{k} \Big|_{p,q} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} x^{m+n+k} \\ &\prod_{u_1=0}^{(r-m)} (p^{u_1} - q^{u_1}x) \prod_{u_2=0}^{(s-n)} (p^{u_2} - q^{u_2}x) \prod_{u_3=0}^{(t-k)} (p^{u_3} - q^{u_3}x). \end{aligned} \tag{1.1}$$

$$f\left(\frac{p^{(r-m)}[m]_{p,q} + p^{(s-n)}[n]_{p,q} + p^{(t-k)}[k]_{p,q}}{[r]_{p,q} + [s]_{p,q} + [t]_{p,q}}\right), x \in [0, 1]$$

Also, we have:

$$\begin{aligned} (1-x)^{rst} &= \sum_{m=0}^r \sum_{n=0}^s \sum_{k=0}^t (-1)^{m+n+k} p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{2}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} \\ &\binom{r}{m} \binom{s}{n} \binom{t}{k} \Big|_{p,q} x^{m+n+k}. \end{aligned}$$

$$\begin{aligned} S_{rst,p,q}(f, x) &= \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}} \sum_{m=0}^r \sum_{n=0}^s \sum_{k=0}^t \binom{r}{m} \binom{s}{n} \binom{t}{k} \Big|_{p,q} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} x^{m+n+k} \\ &\prod_{u_1=0}^{(r-m)} (p^{u_1} - q^{u_1}x) \prod_{u_2=0}^{(s-n)} (p^{u_2} - q^{u_2}x) \prod_{u_3=0}^{(t-k)} (p^{u_3} - q^{u_3}x). \end{aligned} \tag{1.2}$$

$$f\left(\frac{p^{(r-m)}[m]_{p,q} + p^{(s-n)}[n]_{p,q} + p^{(t-k)}[k]_{p,q} + \eta}{[r]_{p,q} + [s]_{p,q} + [t]_{p,q} + \mu}\right), x \in [0, 1]$$

Let us note the fact that for $\eta = \mu = 0$, (p, q) -Bernstein-Stancu operators given by (1.2) converts into (p, q) -Bernstein operators. At the same time for $p = 1$, (p, q) -Bernstein-Stancu operators given by (1.1) turn out to be q -Bernstein-Stancu operators.

Throughout the paper, \mathbb{R} denotes the real of three dimensional space with metric (X, d) . Let us consider a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ such that $(S_{rst,p,q}(f, x)) \in \mathbb{R}, m, n, k \in \mathbb{N}$.

Let f be a given continuous function defined over the closed range interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is said to be statistically convergent to (f, x) , written as $st_3\text{-lim } S_{rst,p,q}(f, x) = f(x)$, provided that

the set:

$$K_\epsilon := \{(m, n, k) \in \mathbb{N}^3 : |S_{rst,p,q}(f, x) - (f, x)| \geq \epsilon\}$$

has natural density zero for any $\epsilon > 0$. In this case, (f, x) is called the statistical limit of the triple sequence of Bernstein-Stancu polynomials. i.e., $\delta_3(K_\epsilon) = 0$. That is:

$$\lim_{r,s,t \rightarrow \infty} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t : |S_{rst,p,q}(f, x) - (f, x)| \geq \epsilon\}| = 0.$$

In this case, we write δ_3 -lim $S_{rst,p,q}(f, x) = (f, x)$ or $S_{rst,p,q}(f, x) \xrightarrow{st3} (f, x)$.

Throughout the paper, \mathbb{N} denotes the set of all positive integers, χ_A - the characteristic function of $A \subset \mathbb{N}$, \mathbb{R} the set of all real numbers. A subset A of \mathbb{N}^3 is said to have asymptotic density $d_3(A)$ if:

$$d_3(A) = \lim_{i,j,\ell \rightarrow \infty} \frac{1}{ij\ell} \sum_{m=1}^i \sum_{n=1}^j \sum_{k=1}^{\ell} \chi_A(K).$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C})$, where \mathbb{N} , \mathbb{R} and \mathbb{C} denote the set of natural numbers, real numbers and complex numbers, respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner *et al.* (2007; Sahiner and Tripathy, 2008), Esi (2014; Esi and Catalbas, 2014; Esi and Savas, 2015; Esi *et al.*, 2016), Dutta *et al.* (2013), Subramanian and Esi (2015), Debnath *et al.* (2015) and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if:

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3 .

A fuzzy number X is a fuzzy subset of the real \mathbb{R}^3 , which is normal fuzzy convex, upper semi-continuous and the X^0 is bounded where $X^0 := \text{cl}\{x \in \mathbb{R}^3 : X(x) > 0\}$ and cl is the closure operator. These properties imply that for each $\alpha \in (0, 1]$, the α -level set X^α defined by:

$$X^\alpha = \{x \in \mathbb{R}^3 : X(x) \geq \alpha\} = [\underline{X}^\alpha, \bar{X}^\alpha]$$

is a non empty compact convex subset of \mathbb{R}^3 .

The supremum metric d on the set $L(\mathbb{R}^3)$ is defined by:

$$d(X, Y) = \sup_{\alpha \in [0,1]} \max(|\underline{X}^\alpha - \underline{Y}^\alpha|, |\bar{X}^\alpha - \bar{Y}^\alpha|).$$

Now, given $X, Y \in L(\mathbb{R}^3)$, we define $X \leq Y$ if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\bar{X}^\alpha \leq \bar{Y}^\alpha$ for each $\alpha \in [0, 1]$.

We write $X \leq Y$ if $X \leq Y$ and there exists an $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} \leq \underline{Y}^{\alpha_0}$ or $\bar{X}^{\alpha_0} \leq \bar{Y}^{\alpha_0}$.

A subset E of $L(\mathbb{R}^3)$ is said to be bounded above if there exists a fuzzy number μ , called an upper bound of E , such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound of E if μ is an upper bound and $\mu \leq \mu'$ for all upper bounds μ' .

A lower bound and the greatest lower bound are defined similarly. E is said to be bounded if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E \subset L(\mathbb{R}^3)$ is bounded then its supremum and infimum exist.

The limit infimum and limit supremum of a triple sequence spaces (X_{mnk}) is defined by:

$$\liminf_{mnk \rightarrow \ell} X_{mnk} := \inf A_X.$$

$$\limsup_{mnk \rightarrow \ell} X_{mnk} := \inf B_X.$$

where:

$$A_X := \{\mu \in L(\mathbb{R}^3) : \text{The set } \{(m, n, k) \in \mathbb{N}^3 : X_{mnk} < \mu\} \text{ is infinite}\}$$

$$B_X := \{\mu \in L(\mathbb{R}^3) : \text{The set } \{(m, n, k) \in \mathbb{N}^3 : X_{mnk} < \mu\} \text{ is infinite}\}.$$

Now, given two fuzzy numbers $X, Y \in L(\mathbb{R}^3)$, we define their sum as $Z = X + Y$, where $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\bar{Z}^\alpha := \bar{X}^\alpha + \bar{Y}^\alpha$ for all $\alpha \in [0, 1]$.

To any real number $a \in \mathbb{R}^3$, we can assign a fuzzy number $a_1 \in L(\mathbb{R}^3)$, which is defied by:

$$a_1(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} \end{cases}$$

An order interval in $L(\mathbb{R}^3)$ is defined by $[X, Y] := \{Z \in L(\mathbb{R}^3) : X \leq Z \leq Y\}$, where $X, Y \in L(\mathbb{R}^3)$.

A set E of fuzzy numbers is called convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in E$ for all $\lambda \in [0, 1]$ and $\mu_1, \mu_2 \in E$.

Definitions and Preliminaries

Throughout the paper \mathbb{R}^3 denotes the real three dimensional case with the metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m, n, k \in \mathbb{N}$. Those definitions follows:

Definition 2.1

Let f be a given continuous function defined over the given closed interval $[0, 1]$. A triple sequence of fuzzy

Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is said to be statistically convergent to (f, X) given by $S_{rst,p,q}(f, X) \rightarrow^{st\text{-lim } x} (f, X)$, if for every $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where:

$$A(\epsilon) = \{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| \geq \epsilon\}.$$

Definition 2.2

Let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is said to be statistically convergent to (f, X) symbolized by $S_{rst,p,q}(f, X) \rightarrow^{st\text{-lim } x} (f, X)$, provided that the set:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| \geq \epsilon\},$$

has natural density zero for every $\epsilon > 0$. In this case, (f, X) is called the statistical limit of the sequence of fuzzy Bernstein-Stancu polynomials.

Definition 2.3

Let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ in a metric space $(X, |.,.|)$ and β be a non-negative real number is said to be β -convergent to (f, X) , symbolized by $S_{rst,p,q}(f, X) \rightarrow^\beta (f, X)$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $r, s, t \geq N_\epsilon$ we have:

$$|S_{rst,p,q}(f, X) - (f, X)| < r + \epsilon$$

Under these conditions $S_{rst,p,q}(f, X)$ is known as an β -limit of (f, X) .

Remark 2.1

We are going to consider β -limit set $S_{rst,p,q}(f, X)$ which is described by $LIM^\beta S_{rst,p,q}(f, X)$ and is defined by:

$$LIM^\beta S_{rst,p,q}(f, X) = \{f : S_{rst,p,q}(f, X) \rightarrow^\beta (f, X)\}.$$

Definition 2.4

Let f be a continuous function defined on the closed interval $[0, 1]$.

A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is said to be β -convergent if $LIM^\beta S_{rst,p,q}(f, X) \neq \phi$ and β is called a rough convergence degree of $S_{rst,p,q}(f, X)$. If $\beta = 0$ then it is ordinary convergence of triple sequence of fuzzy Bernstein-Stancu polynomials.

Definition 2.5

Let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy

Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ in a metric space $(X, |.,.|)$ and β be a non-negative real number is said to be β -statistically convergent to (f, X) , denoted by $S_{rst,p,q}(f, X) \rightarrow^{\beta\text{-st}_3} (f, X)$, if for any $\epsilon > 0$ we conclude that $d_3(A(\epsilon)) = 0$, where:

$$A(\epsilon) = \{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X) - (f, X)| \geq r + \epsilon\}.$$

In this case (f, X) is called β -statistical limit of $S_{rst,p,q}(f, X)$. If $\beta = 0$ then it is ordinary statistical convergent of triple sequence of fuzzy Bernstein-Stancu polynomials.

Definition 2.6

The class denoted by I of subsets of a given nonempty set X is said to be an ideal in X provided:

- (i) $\phi \in I$;
 - (ii) $A, B \in I$ implies $A \cup B \in I$
 - (iii) $A \in I, B \subset A$ implies $B \in I$
- I is called a nontrivial ideal if $X \notin I$.

Definition 2.7

A nonempty class F of subsets of a nonempty set X is said to be a filter in X . Provided:

- (i) $\phi \in F$
- (ii) $A, B \in F$ implies $A \cap B \in F$;
- (iii) $A \in F, A \subset B$ implies $B \in F$.

Definition 2.8

I is a non trivial ideal in $X, X \neq \phi$, then the class:

$$F(I) = \{M \subset X : M = X \setminus A \text{ for some } A \in I\}$$

is a filter on X , called the filter associated with I .

Definition 2.9

A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

Note 2.1

If I is an admissible ideal, then usual convergence in X implies I convergence in X .

Remark 2.2

If I is an admissible ideal, then usual rough convergence implies rough I -convergence.

Definition 2.10

Let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ in a metric

space $(X, |.,.|)$ and β be a non-negative real number is said to be rough ideal convergent or βI -convergent to (f, X) , denoted by $S_{rst,p,q}(f, X) \rightarrow^{\beta I} (f, X)$, if for any $\epsilon > 0$ we have:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| \geq \beta + \epsilon\} \in I.$$

In this case $(S_{rst,p,q}(f, X))$ is called βI -convergent to (f, X) and a triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is called rough I -convergent to (f, X) with β as roughness of degree. If $\beta = 0$ then it is ordinary I -convergent.

Note 2.2

In general, let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(g, X))$ is not I -convergent in usual sense and $|S_{rst,p,q}(f, X) - S_{rst,p,q}(g, X)| \leq \beta$ for all $(r, s, t) \in \mathbb{N}^3$ or:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - S_{rst,p,q}(g, X)| \geq \beta\} \in I$$

for some $\beta > 0$. Under these conditions, the triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is βI -convergent.

Note 2.3

It is obvious that it is not necessary βI -limit of a sequence $S_{rst,p,q}(f, X)$ of fuzzy Bernstein-Stancu polynomial is unique.

Definition 2.11

Let us consider βI -limit set of (f, X) , which is denoted by:

$$I - LIM^\beta S_{rst,p,q}(f, X) = \{f : S_{rst,p,q}(f, X) \rightarrow^{\beta I} (f, X)\},$$

then the triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is said to be βI -convergent if $I - LIM^\beta S_{rst,p,q}(f, X) \neq \emptyset$ and β is called a rough I -convergence degree of $S_{rst,p,q}(f, X)$.

Definition 2.12

Let f be a given continuous function defined on the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is said to be I -analytic if there exists a positive real number M such that:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X)|^{l+m+n+k} \geq M\} \in I.$$

Definition 2.13

Let f be a given continuous function defined over the closed interval $[0, 1]$. A point $L \in X$ is said to be an I -accumulation point of a fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ in a given space having the metric (X, d) iff for each $\epsilon > 0$ the set:

$$\left\{ \begin{aligned} & \{(r, s, t) \in \mathbb{N}^3 : d(S_{rst,p,q}(f, X), (f, X))\} \\ & = \{ |S_{rst,p,q}(f, X) - (f, X)| < \epsilon \} \end{aligned} \right\} \notin I.$$

We denote the set of all I -accumulation points of $(S_{rst,p,q}(f, X))$ by $I(\Gamma(S_{rst,p,q}(f, X)))$.

Definition 2.14

Let f be a given continuous function defined over the closed interval $[0, 1]$. A given triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ is said to be rough I -convergent if $I - LIM^\beta S_{rst,p,q}(f, X) \neq \emptyset$. It is clear that if $I - LIM^\beta S_{rst,p,q}(f, X) \neq \emptyset$ for a triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ of real numbers, then we have:

$$\begin{aligned} & I - LIM^\beta S_{rst,p,q}(f, X) \\ & = [I - \limsup S_{rst,p,q}(f, X) - \beta, I - \liminf S_{rst,p,q}(f, X) + \beta]. \end{aligned}$$

Definition 2.15

Let f be a given continuous function defined over the closed interval $[0, 1]$. The number of $\bar{\beta} = \{\beta \geq 0 : I - LIM^\beta S_{rst,p,q}(f, X) \neq \emptyset\}$ is said to be rough I -core $S_{rst,p,q}(f, X)$ of triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$.

Main Results

Theorem 3.1

Let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ of real numbers and $I \subset 2^{\mathbb{N}}$ be an admissible ideal, we obtain $\text{diam}(I - LIM^\beta S_{rst,p,q}(f, X)) \leq 2\beta$. In general, $\text{diam}(I - LIM^\beta S_{rst,p,q}(f, X))$ has got an upper bound.

Proof

Assume that $\text{diam}(I - LIM^\beta S_{rst,p,q}(f, X))$. Then, $\exists S_{rst,p,q}(f, X)$:

$$\begin{aligned} & S_{rst,p,q}(q, X) \in LIM^\beta S_{rst,p,q}(f, X) \\ & \exists : |S_{rst,p,q}(p, X) - S_{rst,p,q}(q, X)| > 2\beta. \end{aligned}$$

Take $\epsilon \in \left(0, \frac{|S_{rst,p,q}(p, X) - S_{rst,p,q}(q, X)|}{2} - \beta\right)$.

Because:

$$S_{rst,p,q}(p, X), S_{rst,p,q}(q, X) \in I - LIM^\beta S_{rst,p,q}(f, X),$$

we have $A_1(\epsilon) \in I$ and $A_2(\epsilon) \in I$ for each $\epsilon > 0$, in which:

$$A_1(\epsilon) = \{(u, v, w) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - S_{rst,p,q}(p, X)| \geq r + \epsilon\}$$

and:

$$A_2(\epsilon) = \{(u, v, w) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - S_{rst,p,q}(q, X)| \geq r + \epsilon\}.$$

By utilizing the properties $F(I)$, we obtain:

$$(A_1(\epsilon)^c \cap A_2(\epsilon)^c) \in F(I).$$

Therefore we can write:

$$|S_{rst,p,q}(p, X) - S_{rst,p,q}(q, X)| \leq |S_{rst,p,q}(f, X) - S_{rst,p,q}(p, X)| + |S_{rst,p,q}(f, X) - S_{rst,p,q}(q, X)| < (\beta + \epsilon) + (\beta + \epsilon) < 2(\beta + \epsilon),$$

for all $(r, s, t) \in A_1(\epsilon)^c \cap A_2(\epsilon)^c$ which is a contradiction. Hence $\text{diam}(LIM^\beta S_{rst,p,q}(f, X)) \leq 2\beta$.

Now, it is time to take into consideration a triple sequence of fuzzy Bernstein-Stancu polynomials of $(S_{rst,p,q}(f, X))$ of real numbers such that $I\text{-}\lim_{rst \rightarrow \infty} S_{rst,p,q}(f, X) = (f, X)$.

Let $\epsilon > 0$. Then we can write:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| \geq \epsilon\} \in I$$

Therefore we have:

$$\begin{aligned} |S_{rst,p,q}(f, X) - S_{rst,p,q}(p, X)| &\leq |S_{rst,p,q}(f, X) - (f, X)| \\ &+ |(f, X) - S_{rst,p,q}(p, X)| \\ &\leq |S_{rst,p,q}(f, X) - (f, X)| + \beta \leq \beta + \epsilon, \end{aligned}$$

for each:

$$\begin{aligned} S_{rst,p,q}(p, X) &\in \bar{S}_\beta((f, X)): \\ &= \{S_{rst,p,q}(p, X) \in \mathbb{R}^3 : |S_{rst,p,q}(p, X) - (f, X)| \geq \beta\}. \end{aligned}$$

Next we obtain:

$$|S_{rst,p,q}(f, X) - S_{rst,p,q}(p, X)| < \beta + \epsilon$$

for each $(r, s, t) \in \{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| < \epsilon\}$. Because the triple sequence of fuzzy Bernstein-Stancu polynomials of $S_{rst,p,q}(f, X)$ is I -convergent to (f, X) , we have:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| < \epsilon\} \in F(I).$$

Therefore, we obtain $p \in I\text{-}LIM^r S_{rst,p,q}(f, X)$. Consequently, we can write:

$$I - LIM^\beta S_{rst,p,q}(f, X) = \bar{S}_\beta((f, X)). \tag{3.1}$$

Since $\text{diam}(\bar{S}_\beta((f, X))) = 2\beta$, this shows that in general, the upper bound 2β of the diameter of the set $I\text{-}LIM^\beta S_{rst,p,q}(f, X)$ is not lower bound.

Theorem 3.2

Let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ of real numbers, $I \subset 3^{\mathbb{N}}$ be an admissible ideal. For an arbitrary $(f, c) \in I(\Gamma_X)$, we obtain $|S_{rst,p,q}(f, X) - (f, c)| \leq \beta$ for all $S_{rst,p,q}(f, X) \in I\text{-}LIM^\beta S_{rst,p,q}(f, X)$.

Proof

Assume to the contrary that there is a point $(f, c) \in I(\Gamma_X)$ and $S_{rst,p,q}(f, X) \in I\text{-}LIM^\beta S_{rst,p,q}(f, X)$ such that $|S_{rst,p,q}(f, X) - (f, c)| > \beta$. Define

$$\epsilon := \frac{|S_{rst,p,q}(f, X) - (f, c)| - \beta}{3}. \text{ Then:}$$

$$\begin{aligned} &\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, c)| < \epsilon\} \\ &\subseteq \{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| \geq \beta + \epsilon\}. \end{aligned} \tag{3.2}$$

Due to the fact that $(f, c) \in I(\Gamma_X)$, we have:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, c)| < \epsilon\} \notin I.$$

However, using the definition of I -convergence, because:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| \geq \beta + \epsilon\} \in I,$$

so by (3.2) we have:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, c)| < \epsilon\} \in I,$$

which is a contradiction to the fact $(f, c) \in I(\Gamma_X)$. On the other hand, if $(f, c) \in I(\Gamma_X)$ i.e.:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, c)| < \epsilon\} \notin I,$$

then:

$$\{(r, s, t) \in \mathbb{N}^3 : |S_{rst,p,q}(f, X) - (f, X)| \geq \beta + \epsilon\} \notin I,$$

which is a contradiction to the fact that $(f, X) \in I\text{-LIM}^\beta S_{rst,p,q}(f, X)$.

Theorem 3.3

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials:

$$\begin{aligned} (S_{rst,p,q}(f, X)) &\rightarrow^I (f, X) \\ \Leftrightarrow I\text{-LIM}^\beta S_{rst,p,q}(f, X) &= \bar{S}_\beta((f, X)). \end{aligned}$$

Proof

Necessity: By Theorem 3.1.

Sufficiency

Assume that $I\text{-LIM}^\beta S_{rst,p,q}(f, X) = \bar{S}_\beta((f, X)) (\neq \phi)$. Thus the triple sequence spaces of fuzzy Bernstein-Stancu polynomials of $(S_{rst,p,q}(f, X))$ is I -analytic. If we assume that (f, X) has another I -cluster point (f', X) different from (f, X) . Then the following point:

$$\begin{aligned} (\bar{f}, X) &= (f, X) + \frac{\beta}{|(f, X) - (f', X)|} ((f, X) - (f', X)) \\ (\bar{f}, X) - (f', X) &= (f, X) - (f', X) \\ &+ \frac{\beta}{|(f, X) - (f', X)|} [((f, X) - (f', X)) - ((f', X) - (f', X))] \\ |(\bar{f}, X) - (f', X)| &= |(f, X) - (f', X)| \\ &+ \frac{\beta}{|(f, X) - (f', X)|} |(f, X) - (f', X)| \\ |(\bar{f}, X) - (f', X)| &= |(f, X) - (f', X)| + \beta > \beta. \end{aligned}$$

Since $(f', X) \in I(\Gamma_X)$, using Theorem 3.2, $(\bar{f}, X) \notin I\text{-LIM}^\beta S_{rst,p,q}(f, X)$. It is not possible as $|(\bar{f}, X) - (f, X)| = \beta$ and $I\text{-LIM}^\beta S_{rst,p,q}(f, X) = \bar{S}_\beta((f, X))$. Since (f, X) is the unique- I -cluster point of (f, X) . Thus $\Rightarrow S_{rst,p,q}(f, X) \rightarrow^I (f, X)$.

Corollary 3.1

If $(X, |.,.>)$ is a strictly convex spaces and at the same time f is a given continuous function given over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X)) \in X$, there

exists $y_1, y_2 \in I\text{-LIM}^\beta S_{rst,p,q}(f, X)$ such that $|y_1 - y_2| = 2\beta$, then this triple sequence of fuzzy $(f, X) \rightarrow^I \frac{y_1 + y_2}{2}$.

Proof

Omitted.

Theorem 3.4

If $I\text{-LIM}^\beta \neq \phi$, then $I\text{-lim sup } S_{rst,p,q}(f, X)$ and $I\text{-lim inf } S_{rst,p,q}(f, X)$ are in the set $I\text{-LIM}^{2\beta} S_{rst,p,q}(f, X)$.

Proof

We know that $I\text{-LIM}^\beta S_{rst,p,q}(f, X) \neq \phi$, a triple sequence of fuzzy Bernstein-Stancu polynomials of $(S_{rst,p,q}(f, X))$ is I -analytic. The number $I\text{-lim inf } S_{rst,p,q}(f, X)$ is an I -cluster point of (f, X) and thus, we obtain:

$$\begin{aligned} |(f, X) - I\text{-lim inf } S_{rst,p,q}(f, X)| \\ \leq \beta \forall (f, X) \in I\text{-LIM}^\beta(f, X). \end{aligned}$$

Let $A = \{(r, s, t) \in \mathbb{N}^3 : |(f, X) - S_{rst,p,q}(f, X)| \geq \beta + \epsilon\}$. Now if $(r, s, t) \notin A$, then:

$$\begin{aligned} |S_{rst,p,q}(f, X) - (I\text{-lim inf } S_{rst,p,q}(f, X))| &\leq |S_{rst,p,q}(f, X) - (f, X)| \\ &+ |(f, X) - (I\text{-lim inf } S_{rst,p,q}(f, X))| < 2\beta + \epsilon. \end{aligned}$$

Thus:

$$I\text{-lim inf } S_{rst,p,q}(f, X) \in I\text{-LIM}^{2\beta} S_{rst,p,q}(f, X).$$

In a similar way, it can be shown that $I\text{-lim sup } X_{mnk} \in I\text{-LIM}^{2\beta} X_{mnk}$.

Corollary 3.2

Let f be a given continuous function defined on the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f, X))$ of real numbers, if $I\text{-LIM}^\beta S_{rst,p,q}(f, X) \neq \phi$, then:

$$I\text{-core}\{(f, X)\} \subseteq I\text{-LIM}^{2\beta} S_{rst,p,q}(f, X).$$

Proof

We obtain:

$$\begin{aligned} I\text{-LIM}^\beta S_{rst,p,q}(f, X) \\ = [I\text{-lim sup } S_{rst,p,q}(f, X) - 2\beta, I\text{-lim inf } S_{rst,p,q}(f, X) + 2\beta]. \end{aligned}$$

Under these conditions, the result yield in from Theorem 3.4.

Theorem 3.5

Let f be a given continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f,X))$ of real numbers. Then the diam $(I\text{-core}\{S_{rst,p,q}(f,X)\})$ of the set:

$$I\text{-core}\{S_{rst,p,q}(f,X)\} = \beta \Leftrightarrow I\text{-core}\{(f,X)\} = I\text{-LIM}^\beta S_{rst,p,q}(f,X)$$

Proof:

$$\begin{aligned} \text{diam}(I\text{-core}\{S_{rst,p,q}(f,X)\}) &= \beta \\ \Leftrightarrow (I\text{-limsup} S_{rst,p,q}(f,X)) - (I\text{-limsup} X_{mnk}) &= \beta \\ \Leftrightarrow I\text{-core}\{X_{mnk}\} &= [I\text{-liminf} X_{mnk}, I\text{-limsup} S_{rst,p,q}(f,X)] \\ &= [I - I\text{-limsup} S_{rst,p,q}(f,X) - \beta, I\text{-liminf} S_{rst,p,q}(f,X) + r] \\ &= I\text{-LIM}^\beta S_{rst,p,q}(f,X). \end{aligned}$$

Also it is easy to see that:

- (i) $\beta > \text{diam}(I\text{-core}\{S_{rst,p,q}(f,X)\}) \Leftrightarrow I\text{-core}\{S_{rst,p,q}(f,X)\} \subset I\text{-LIM}^\beta S_{rst,p,q}(f,X)$,
- (ii) $\beta > \text{diam}(I\text{-core}\{S_{rst,p,q}(f,X)\}) \Leftrightarrow I\text{-LIM}^\beta S_{rst,p,q}(f,X) \subset I\text{-core}\{S_{rst,p,q}(f,X)\}$

Theorem 3.6

Let f be a continuous function defined over the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f,X))$ of real numbers, if $\bar{\beta} = \inf\{\beta \geq 0: I\text{-LIM}^\beta S_{rst,p,q}(f,X) \neq \phi\}$, then $\bar{\beta} = \text{radius}(I\text{-core}\{S_{rst,p,q}(f,X)\})$.

Proof

If the set $I\text{-core}\{S_{rst,p,q}(f,X)\}$ is singleton, then $\text{radius}(I\text{-core}\{S_{rst,p,q}(f,X)\}) = 0$ and the triple sequence of fuzzy Bernstein-Stancu polynomials is I-convergent, i.e., $I\text{-LIM}^0 S_{rst,p,q}(f,X) \neq \phi$. Hence we get $\bar{\beta} = \text{radius}(I\text{-core}\{S_{rst,p,q}(f,X)\}) = 0$.

Now assume that the set $I\text{-core}\{S_{rst,p,q}(f,X)\}$ is not a singleton. We may write $I\text{-core}\{S_{rst,p,q}(f,X)\} = [a, b]$, where $a = I\text{-lim inf} S_{rst,p,q}(f,X)$ and $b = I\text{-lim sup} S_{rst,p,q}(f,X)$.

Now assume that $\bar{\beta} \neq \text{radius}(I\text{-core}\{S_{rst,p,q}(f,X)\})$.

If $\bar{\beta} < \text{radius}(I\text{-core}\{X_{mnk}\})$, then define $\bar{\epsilon} = \frac{b-a-\bar{\beta}}{3}$. Now, be definition of $\bar{\beta}$ implies that $I\text{-LIM}^{\bar{\beta}+\bar{\epsilon}} S_{rst,p,q}(f,X) \neq \phi$, given $\epsilon > 0$:

$$\exists I \in \mathbb{R} : A \left\{ \begin{aligned} &(r,s,t) \in \mathbb{N} : S_{rst,p,q}(f,X) \\ &-(f,X) \geq (\bar{\beta} + \bar{\epsilon}) + \epsilon \end{aligned} \right\} \in I.$$

Since $\bar{\beta} + \bar{\epsilon} < \frac{b-a}{2}$ which is a contradiction of the definition of a and b .

If $\bar{\beta} > \text{radius}(I\text{-core}\{S_{rst,p,q}(f,X)\})$, then define $\bar{\epsilon} = \frac{\bar{\beta} - \frac{b-a}{2}}{3}$ and $\beta' = \bar{\beta} - 2\bar{\epsilon}$.

It is clear that $0 \leq \beta' \leq \bar{\beta}$ and using the descriptions of a and b , the number $\frac{b-a}{2} \in I\text{-LIM}^{\beta'} S_{rst,p,q}(f,X)$. Under these conditions, we obtain:

$$\bar{\beta} \in \{\beta \geq 0 : I\text{-LIM}^\beta S_{rst,p,q}(f,X) \neq \phi\},$$

which is a contradiction to the equality:

$$\bar{\beta} = \inf\{\beta \geq 0 : I\text{-LIM}^\beta S_{rst,p,q}(f,X) \neq \phi\} \text{ as } \beta' < \bar{\beta}.$$

Corollary 3.3

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of fuzzy Bernstein-Stancu polynomials $(S_{rst,p,q}(f,X))$ of real numbers, then $I\text{-core}\{S_{rst,p,q}(f,X)\} = I\text{-LIM}^{2\bar{\beta}} S_{rst,p,q}(f,X)$.

Proof

It follows that Theorem 3.5 and Theorem 3.6.

Example 3.1

Let f be a continuous function defined on the closed interval $[0, 1]$.

With the help of Matlab, we show comparisons and some illustrative figures on the convergence of operators given in (1.2) to the function $f(X) = 1 + X^3 \sin(14X)$ under different arguments.

From Fig. 1a, we can see that as the value the q and p goes to 1 under the conditions $0 < q < p \leq 1$, (p, q) -Bernstein-Stancu operators given by (1.2) converges to the function $f(X) = 1 + X^3 \sin(14X)$. From these figures, namely 1(a) and (b), we can see that for $\eta = \mu = 0$, when the values of the rst go up, (p, q) -Bernstein-Stancu operators given by (1.2) converges to the function. In a similar manner, using the Fig. 2a, we may conclude that for $\eta = \mu = 5$, when the values of the q and p go to 1 or else the condition $0 < q < p \leq 1$ is provided, (p, q) -Bernstein-Stancu operators given by (1.2) converges towards the function. Using the Fig. 2a and 2b, we can conclude that as the values of the $[rst]$ go up, (p, q) -Bernstein-Stancu operators given by $f(X) = 1 + X^3 \sin(14X)$ converges to the function.

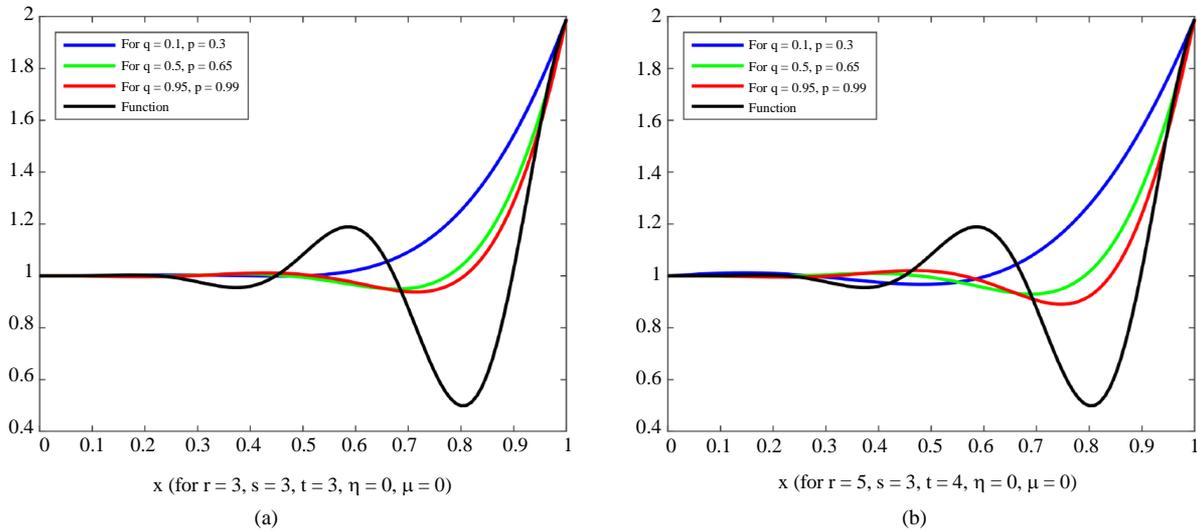


Fig. 1: (p, q) -Bernstein-Stancu operators

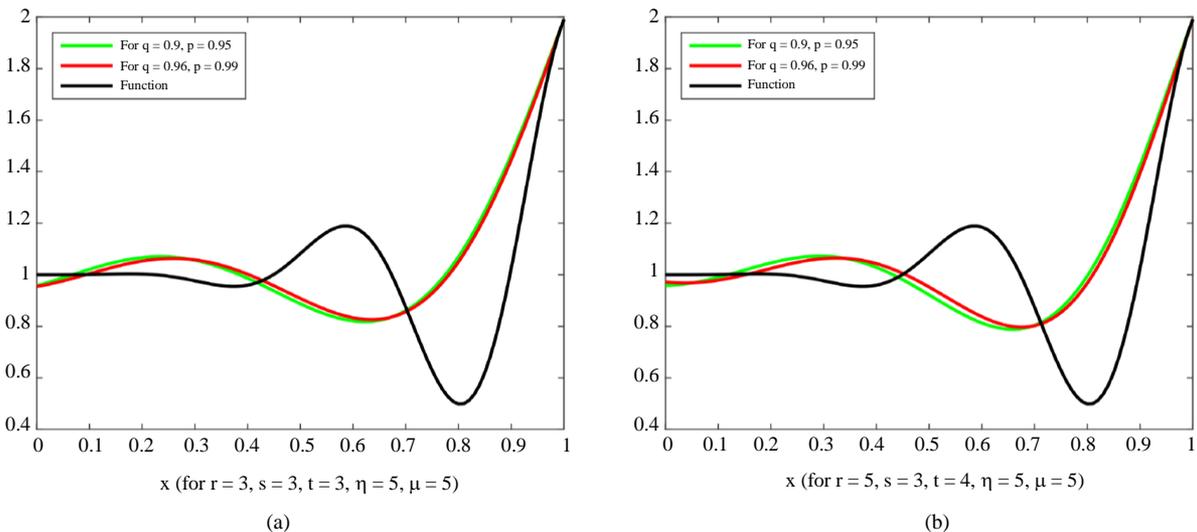


Fig. 2: (p, q) -Bernstein-Stancu operators

Conclusion

We introduced and studied some basic properties of fuzzy Bernstein-Stancu operators of rough I-convergent of triple sequences and also studied the set of all fuzzy Bernstein-Stancu operators of rough I-limits of a triple sequence and relation between analytic ness and Fuzzy Bernstein-Stancu operators of rough I-core of a triple sequence.

For the reference sections, considered the following introduction described the main results are motivating the research.

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Author's Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing Interests

The authors of the present manuscript declare that there is not any conflict of interests regarding the publication of this study.

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