

# Study of Fractional Order Integro-Differential Equations by using Chebyshev Neural Network

<sup>1</sup>Sarkhosh S. Chaharborj, <sup>2</sup>Shahriar S. Chaharborj and <sup>3</sup>Yaghoob Mahmoudi

<sup>1</sup>School of Mathematics and Statistics, Carleton University, Ottawa, Canada

<sup>2</sup>Department of Mathematics, Islamic Azad University, Bushehr Branch, Bushehr, Iran

<sup>3</sup>Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran

## Article history

Received: 11-04-2016

Revised: 01-02-2017

Accepted: 06-02-2017

Corresponding Author:  
Sarkhosh S. Chaharborj  
School of Mathematics and  
Statistics, Carleton University,  
Ottawa, Canada  
Email: sseddighi2014@yahoo.com.my

**Abstract:** Recently fractional integro-differential equations have been proposed for the modeling of many complex systems. Simulation of these equations requires the development of suitable numerical methods. This paper is devoted to finding the numerical solution of some classes of fractional integro-differential equations by employing the Chebyshev Neural Network (ChNN). The accuracy of proposed method is shown by comparing the numerical results computed by using Chebyshev neural network method with the analytical solution.

**Keywords:** Integro-Differential Equations, Fractional Order, Chebyshev Neural Network, Error Back Propagation Algorithm, Feed Forward Neural Network

## Introduction

Numerical methods are very power tools for solving the complicated problems in many fields (Bianca *et al.*, 2009; Bhrawy and Alghamdi, 2012; Yang *et al.*, 2014; Bhrawy and Aloji, 2013; Doha *et al.*, 2011; Irandoust-Pakchin *et al.*, 2013; Irandoust-Pakchin and Abdi-Mazraeh, 2013; Saha Ray, 2009; Mittal and Nigam, 2008; Saeedi and Samimi, 2012; Saeed and Sdeq, 2010; Ahmed and Salh, 2011). Newly, few numerical methods for solving the Fractional Differential Equations (FDEs) and Fractional Integro-Differential Equations (FIDEs) have been presented. Bhrawy and Alghamdi (2012; Yang *et al.*, 2004) used collocation method to solve the nonlinear fractional Langevin equation involving two fractional orders in different intervals and the fractional Fredholm Integro-differential equations. Doha *et al.* (2011; Bhrawy and Aloji, 2003; Irandoust-pakchin *et al.*, 2013), proposed the Chebyshev polynomials method to solve the nonlinear Volterra and the Fredholm Integro-differential equations of fractional order and the multiterm fractional orders differential equations. Irandoust-Pakchin and Abdi-Mazraeh (2013) applied the variational iteration method to solve the fractional Integro-differential equations with the nonlocal boundary conditions. Few other methods presented for solving the fractional diffusion; the fractional Integro-differential and the fractional nonlinear Fredholm

Integro-differential equations in (Saha Ray, 2009; Mittal and Nigam, 2008; Saeedi and Samimi, 2012; Saeed and Sdeq, 2010).

In this study, Chebyshev neural network method with a unit layer has been proposed to solve the integro-differential equations from fractional order. To minimize the computed error function, a neural network with feed forward and with fundamental error back propagation will be used. This paper deals with numerical analysis of fractional order integro-differential equation as follows:

$$D^\alpha y(x) = g(x) + \int_a^x f_1(x, \tau) y(\tau) d\tau + \int_a^b f_2(x, \tau) y(\tau) d\tau, \quad a \leq x, \tau \leq b \quad (1)$$

with the following initial conditions:

$$y^{(i)}(0) = \beta_i, \quad n-1 < \alpha \leq n, \quad n \in N$$

The problem of the nonlinear multi-order fractional differential equations studies by using the Chebyshev neural network to obtain the numerical results. The hidden layer was excluded by extending the input style by the Chebyshev polynomials and has been used a single layer neural network. The idea of the method is to find the semi-analytical solution of this kind equations with high accuracy.

In section 2, basic definitions of the fractional derivatives will be presented. In section 3, main results of proposed method including learning algorithm of the Chebyshev neural network, Chebyshev neural network formulation for the fractional integro-differential equations, computation of gradient for the Chebyshev neural network and algorithm of the proposed method will be presented. Applications of the proposed method will be shown in Section 4 with solving few examples. Finally we end the paper with some conclusion in section 5.

### Basic Definitions of the Fractional Derivatives

Few definitions of the fractional derivative of order  $\alpha > 0$  can find (Miller and Ross, 1993). The Riemann-Liouville and Caputo fractional derivative are most commonly definitions that we will use in this study. Also, the Riemann-Liouville fractional integration of order  $\alpha$  which using in this research is defined as follows (Grigorenko and Grigorenko, 2003; Podlobny, 2002):

$$J^\alpha h(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} h(\tau) d\tau, \quad \alpha > 0, x > 0 \quad (2)$$

In Equation 2  $\Gamma$  is gamma function and we have:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (3)$$

Caputo and Riemann-Liouville fractional derivatives of order  $\alpha$  (Fadi *et al.*, 2011), respectively can be define:

$$D^\alpha h(x) = \frac{d^n}{dx^n} (J^{n-\alpha} h(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} h^{(n)}(\tau) d\tau; n-1 < \alpha \leq n, n \in N \quad (4)$$

$$D_*^\alpha h(x) = J^{n-\alpha} \left( \frac{d^n}{dx^n} h(x) \right) \quad (5)$$

where,  $D_*^\alpha$  is the Caputo fractional derivative. The properties of the operator  $J^\alpha$  are defined:

$$J^\alpha J^\beta h(x) = J^{\alpha+\beta} h(x),$$

$$J^\alpha J^\beta h(x) = J^\beta J^\alpha h(x), \quad ; \alpha, \beta \geq 0 \text{ and } m \geq -1 \quad (6)$$

$$J^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} x^{\alpha+m}$$

Fundamental property of the Caputo's fractional derivative are shown as follows (Fadi *et al.*, 2011):

$$(J^\alpha D^\beta)h(x) = h(x) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{x^k}{k!}; \quad n-1 < \alpha \leq n, n \in N \quad (7)$$

$$D^\alpha [h(x)g(x)] = \sum_{k=0}^\infty \binom{\alpha}{k} g^{(k)}(x) D^{\alpha-k} h(x) \quad (8)$$

### Main Results

#### Learning Algorithm of the Chebyshev Neural Network

Figure 1 shows the structure of proposed Chebyshev neural network which consists of the unit input block, one output block and a functional extension block based on the Chebyshev polynomials. In this network input data is extended to several terms using Chebyshev polynomials. The learning algorithm can be used to update the network parameters and minimizing the error function. Functions  $F(z) = z$ ;  $\sinh(z)$ ;  $\tanh(z)$  are considered as activation functions. The network output with input data  $x$  and weights  $w$  may be computed as (Mall and Chakraverty, 2014; 2015):

$$N(x; w) = F(z) \quad (9)$$

where,  $z$  is a weighted sum of expanded input data:

$$z = \sum_{j=1}^m w_j T_{j-1}(x) \quad (10)$$

where,  $x$  is the input data,  $T_{j-1}(x)$  and  $w_j$  with  $j = 1, 2, \dots, m$  denote the expanded input data and weight vector, respectively. Two first Chebyshev polynomials are,  $T_0(x) = 1$ ,  $T_1(x) = x$ . The higher order Chebyshev polynomials can be computed by,  $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ , where  $T_n(x)$  denotes  $n$ th order Chebyshev polynomial. Here  $n$  dimensional input pattern is expanded to  $m$  dimensional enhanced Chebyshev polynomials.

Now, weights of proposed network may be modified by using the fundamental back propagation (Mall and Chakraverty, 2014; 2015):

$$w_j^{k+1} = w_j^k + \Delta w_j^k = w_j^k + \left( -\eta \left[ \frac{\partial E(x; w)}{\partial w_j} \right]^k \right) \quad (11)$$

where,  $\eta$  and  $k$  are the learning parameter and iteration step, respectively; which is used to update the weights and  $E(x; w)$  is the error function.

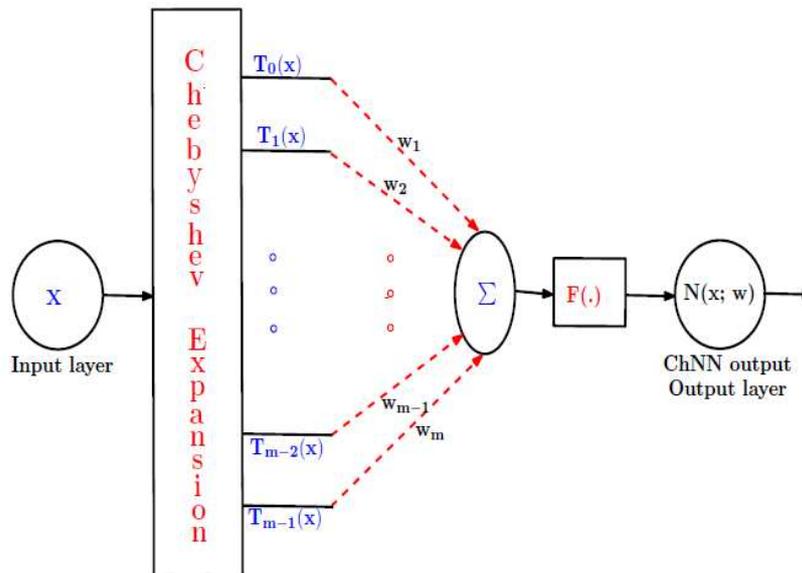


Fig. 1. Proposed Chebyshev Neural Network Structure

*Chebyshev Neural Network Formulation for the Fractional Integro-Differential Equations*

A general form of the fractional integro-differential equations can be shown as follows:

$$\Psi[x, y(x), \nabla y(x), \nabla^2 y(x), \dots, \nabla^n y(x)] = 0, \quad x \in \bar{D} \subseteq R^n \quad (12)$$

where,  $\Psi$  defines the structure of fractional integro-differential equations,  $y(x)$  and  $r$  mean the solution and differential operator, respectively. If  $y_i(x, w)$  indicate the trial solution with variable parameters  $w$ , then from Equation 12 we will have:

$$\Psi \left[ \begin{matrix} x, y_i(x, w), \nabla y_i(x, w), \\ \nabla^2 y_i(x, w), \dots, \nabla^n y_i(x, w) \end{matrix} \right] = 0 \quad (13)$$

From Equation 13, the following minimization equation can conclude (Mall and Chakraverty, 2014; 2015; Hoda and Nagla, 2011):

$$\min_w \frac{1}{2} \left[ \sum_{x \in \bar{D}} \left[ \Psi \left( \begin{matrix} x, y_i(x, w), \nabla y_i(x, w), \\ \nabla^2 y_i(x, w), \dots, \nabla^n y_i(x, w) \end{matrix} \right) \right]^2 \right] \quad (14)$$

Now, we note the general form of fractional order integro-differential equations as shown:

$$D^\alpha y(x) = F(x, y(x), \int_0^x K(x, \tau, y(\tau)) d\tau), \quad 0 < x < 1, 0 < \alpha \leq 1 \quad (15)$$

with the boundary condition,  $y(0) = \gamma_0$ . The trial solution  $y_i(x, w)$  of feed forward neural network with input  $x$  and variable parameters  $w$  for above equation is written as:

$$y_i(x, w) = \gamma_0 + xN(x; w) \quad (16)$$

where,  $N(x; w) = z; \sinh(z); \tanh(z)$ . The error function for the Equation 15 in general form can be shown as (Mall and Chakraverty, 2014; 2015):

$$E(x; w) = \sum_{i=1}^n \frac{1}{2} \left[ \begin{matrix} D^\alpha y_i(x_i, w) \\ -F(x, y_i(x, w), \int_0^x K(x, \tau, y_i(\tau, w)) d\tau) \end{matrix} \right]^2 \quad (17)$$

To minimize the error function  $E(x; w)$  respect to  $w$  and corresponding to input  $x$ , we differentiate  $E(x; w)$  with respect to the parameters  $w$ .

*Gradient Computation for the Proposed Algorithm*

The fractional gradient of proposed network output with respect to network inputs when  $N(x; w) = z; \sinh(z); \tanh(z)$  is computed as below. Fractional derivatives of  $N(x; w)$  can be define as:

$$D^\alpha N(x; w) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} N^{(n)}(\tau; w) d\tau, \quad x > 0, n-1 < \alpha \leq n \quad (18)$$

where,  $\alpha > 0$  is order of fractional derivative and  $n \in N$  is smallest integer greater than  $\alpha$ .

If  $N(x;w) = z$

Fractional derivatives of  $N(x;w) = z$  with respect to input  $x$  is as follows:

$$D^\alpha N(x;w) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha} \sum_{j=1}^M w_j T'_{j-1}(\tau) d\tau, \quad x > 0, 0 < \alpha \leq 1 \quad (19)$$

If  $N(x;w) = \sinh(z)$

Fractional derivatives of  $N(x;w) = \sinh(z)$  with respect to input  $x$  is as follows:

$$D^\alpha N(x;w) = \int_0^x \frac{(x-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \left[ \frac{\cosh\left(\sum_{j=1}^M w_j T'_{j-1}(\tau)\right)}{\sum_{j=1}^M w_j T'_{j-1}(\tau)} \right] d\tau, \quad x > 0, 0 < \alpha \leq 1 \quad (20)$$

If  $N(x;w) = \tanh(z)$

Fractional derivatives of  $N(x;w) = \tanh(z)$  with respect to input  $x$  is as follows:

$$D^\alpha N(x;w) = \int_0^x \frac{(x-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \left[ \frac{1 - \left(\tanh\left(\sum_{j=1}^M w_j T'_{j-1}(\tau)\right)\right)^2}{\sum_{j=1}^M w_j T'_{j-1}(\tau)} \right] d\tau, \quad x > 0, 0 < \alpha \leq 1 \quad (21)$$

where,  $w_j$  denote parameters of network and  $T'_{j-1}$  denote first derivative of the Chebyshev polynomials. From Equation 16 we have:

$$D^\alpha y_i(x;w) D^\alpha [\gamma_0 + xN(\tau;w)], \\ = \frac{1}{\Gamma(1-\alpha)} \int_0^x \left[ \frac{N(\tau;w) + \tau \frac{d}{d\tau} N(\tau;w)}{(x-\tau)^\alpha} \right] d\tau$$

Also, from Equation 17 we have:

$$\frac{\partial E(x;w)}{\partial w_j} = \frac{\partial}{\partial w_j} \left[ \sum_{i=1}^n \frac{1}{2} \left( D^\alpha y_i(x_i;w) - F\left(x, y(x;w), \int_0^x K(x,\tau, y(\tau;w)) d\tau \right) \right)^2 \right] \quad (22)$$

In end, approximate solutions can be computed by using the converged proposed Chebyshev neural network results in Equation 16.

## Applications

In this section, we consider the fractional order integro-differential equations to show the powerfulness of the proposed method. Active functions updated as,  $F(z) = z; \sinh(z); \tanh(z)$  are considered to find the numerical results with high accuracy. The first five Chebyshev polynomials have been used.

### Algorithm 1 Chebyshev Neural Network Algorithm

- Calculating the Chebyshev polynomials  $T_0, T_1, \dots, T_m$
- Replacing the Chebyshev polynomials in the Equation 10
- Obtained  $z$  in the step 2 Can be replaced in the equations  $N(x;w) = z; \sinh(z); \tanh(z)$
- Obtained  $N(x;w)$  in step 3 Can be replaced in the training answer  $y_i(x;w)$
- In continue, training answer  $y_i(x;w)$  will be replaced instead  $y(x)$  in the fractional differential equation.
- Now, interval  $[a, b]$  can be partitioned to  $n$  equal parts with arbitrary distance  $h$ . All values for the fractional differential equation in corresponding points can be defined as,  $E_0, E_1, \dots, E_n$ , respectively
- Error function  $E$  can be defined as,  $E(x;w) = \frac{1}{2} \sum_{i=1}^n E_i^2$ . This error function must to minimize respect to unknown weights  $w_1, w_2, \dots, w_n$
- To minimize the error function  $E$  respect to unknown weights, Gradient of  $E$  respect to  $w_1, w_2, \dots, w_n$  will be used as,  $\frac{\partial E(x;w)}{\partial w_j} = 0$ . This give us a system with  $n$  equations and  $n$  unknowns
- Solving the obtained system by using matrix method or numerical methods as Genetic Algorithm, Bee Colony Optimization Algorithm, Ant Colony Optimization Algorithm, when  $N(x;w)$  is linear or nonlinear, respectively
- The weights may be modified by using the back propagation principle Equation 11
- After replacing modified weights in the training answer  $y_i(x;w)$ , can get approximate solutions for the mentioned equations in this study

### Example 1

First we consider the following fractional integro-differential equation (Fadi *et al.*, 2011):

$$D^{0.5} y(t) = y(t) + \frac{8}{3\Gamma(0.5)} t^{1.5} - t^2 - \frac{1}{3} t^3 + \int_0^t y(\tau) d\tau, \quad t \in [0,1], y(0) = 0 \quad (23)$$

The exact solution of the above equation is given as:

$$y(t) = t^2 \tag{24}$$

The proposed network was trained for ten points in interval  $[0, 1]$  with the first five Chebyshev polynomials ( $m = 5$ ). Comparison of absolute and maximum absolute errors between exact and ChNN solutions with  $F(z) = z$ ;  $\sinh(z)$ ;  $\tanh(z)$  are cited in Table 1 and 2, respectively. Figure 2 shows comparison between analytical and ChNN solutions when the active function is as,  $F(z) = z$ . Plot of the absolute error between analytical and ChNN results with  $F(z) = z$  is shown in Fig. 3. Comparison between analytical and ChNN solutions and absolute error between them with  $F(z) = \sinh(z)$  are showed in Fig. 4 and 5, respectively. Numerical results between analytical and ChNN solutions and absolute error between them with  $F(z) = \tanh(z)$  are cited in Fig. 6 and

7, respectively. According Table 2 the maximum absolute error for active functions  $F(z) = z$ ;  $\sinh(z)$ ;  $\tanh(z)$  are as  $3.40000 \times 10^{-40}$ ,  $0.13542 \times 10^{-2}$  and  $0.40843 \times 10^{-1}$ , respectively. Therefore, the analytical and ChNN solutions have good agreement with the active function  $F(z) = z$ .

*Example 2*

We consider the following fractional integro-differential equation (Fadi *et al.*, 2011):

$$D^{0.75}y(t) = \frac{t^{0.25}}{\Gamma(1.25)} + [t \cos(t) - \sin(t)]y(t) + \int_0^t t \sin(\tau)y(\tau) d\tau, t \in [0,1], y(0) = 0 \tag{25}$$

Table 1. Comparison of absolute errors between the exact and ChNN solutions when  $F(z) = z$ ;  $\sinh(z)$ ;  $\tanh(z)$  (Example 1)

F(z)	t									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
z	[1.9783	1.2490	0.5480	0.8000	1.7600	2.6800	2.9300	2.6300	3.4000]	$\times 10^{-40}$
$\sinh(z)$	[1.1111	0.1096	0.2494	0.5286	0.8353	0.05314	0.6717	0.5634	1.3542]	$\times 10^{-3}$
$\tanh(z)$	[0.02331	0.01347	0.00713	0.06345	0.2544	0.2544	0.77492	1.9234	4.0843]	$\times 10^{-2}$

Table 2. Comparison of maximum absolute errors between the exact and ChNN solutions when  $F(z) = z$ ;  $\sinh(z)$ ;  $\tanh(z)$  (Example 1)

F(z)	Z	$\sinh(z)$	$\tanh(z)$
Max Error	$3.40000 \times 10^{-40}$	$0.13542 \times 10^{-2}$	$0.40843 \times 10^{-1}$

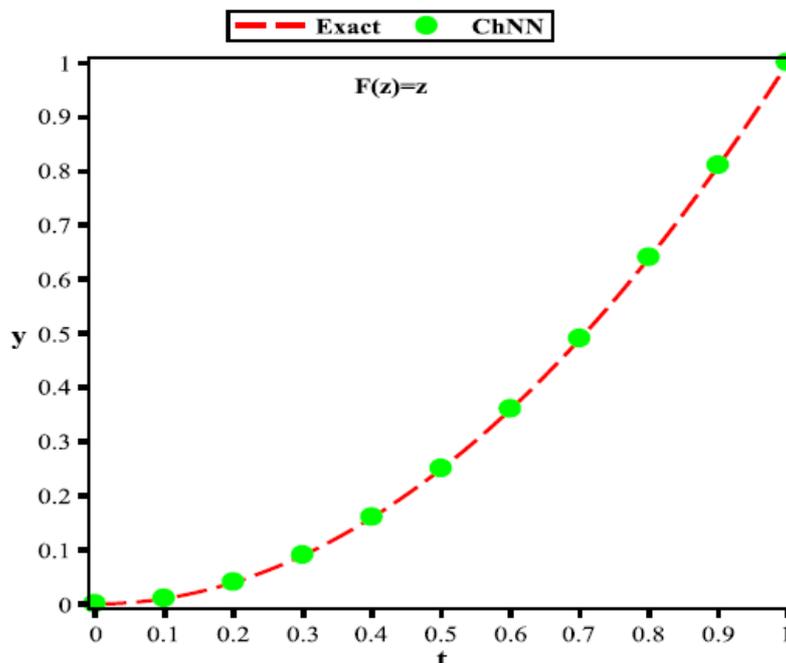


Fig. 2. Numerical results of the exact and ChNN solutions when  $F(z) = z$  (Example 1)

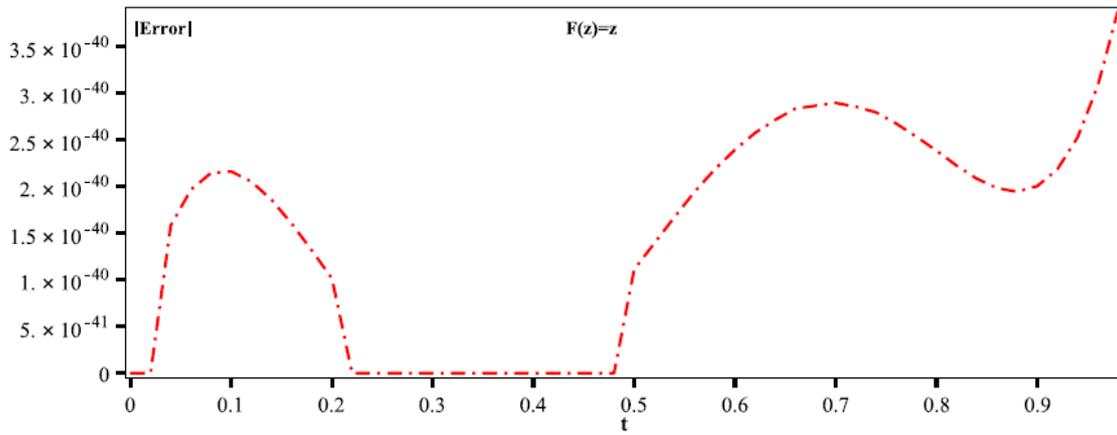


Fig. 3. Plot of absolute error between the exact and ChNN solutions when  $F(z) = z$  (Example 1)

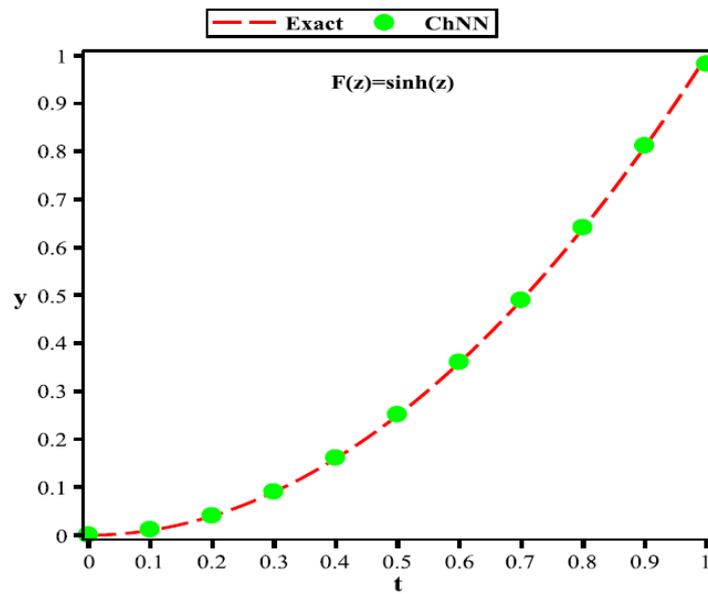


Fig. 4. Numerical results of the exact and ChNN solutions when  $F(z) = \sinh(z)$  (Example 1)

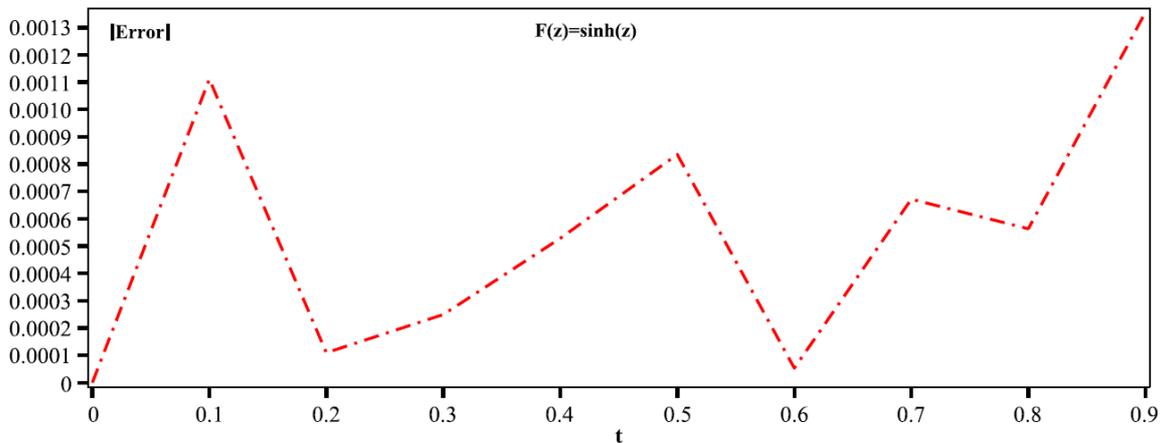


Fig. 5. Plot of absolute error between Exact and ChNN solutions when  $F(z) = \sinh(z)$  (Example 1)

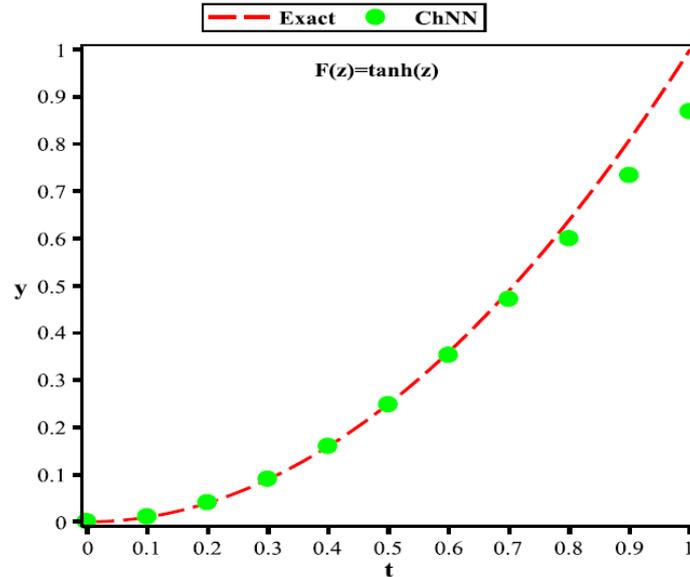


Fig. 6. Numerical results of the exact and ChNN solutions when  $F(z) = \tanh(z)$  (Example 1)

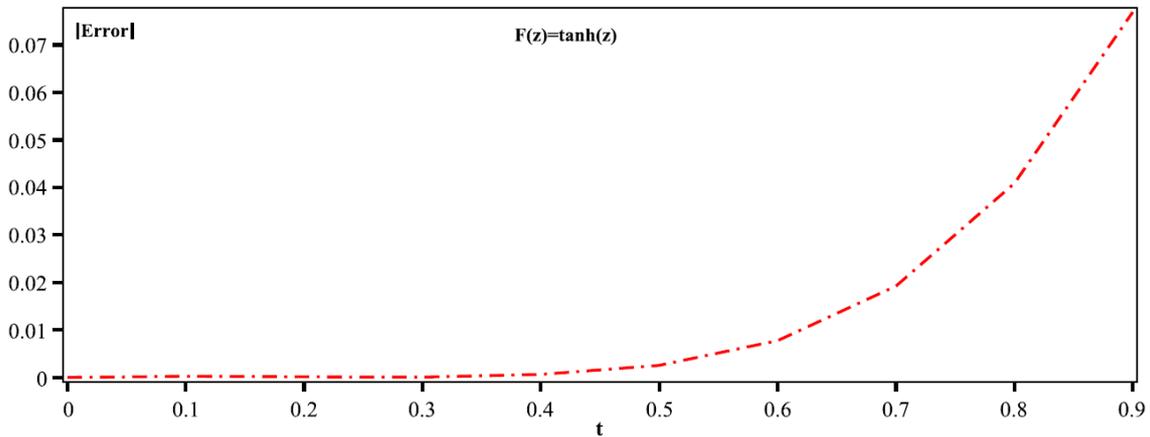


Fig. 7. Plot of absolute error between the exact and ChNN solutions when  $F(z) = \tanh(z)$  (Example 1)

The exact solution of the above equation is given as:

$$y(t) = t \quad (26)$$

Comparison between analytical and ChNN solutions with the active function as,  $F(z) = z$  is cited in Fig. 8. Plot of absolute error between analytical and ChNN solutions with  $F(z) = z$  is shown in Fig. 9.

### Example 3

Consider the following fractional integro-differential equation (Hasan *et al.*, 2013):

$$D^{0.5}y(t) = \frac{8/3t^{3/2} - 2t^{1/2}}{\sqrt{\pi}} + \frac{t}{12} + \int_0^t t(\tau^3 - \tau^2)d\tau, t \in [0,1], y(0) = 0 \quad (27)$$

with the exact solution as:

$$y(t) = t^2 - t \quad (28)$$

Figure 10 shows comparison between analytical and ChNN solutions with the active function as,  $F(z) = z$ . Figure 11 presents plot of absolute error between analytical and ChNN solutions with  $F(z) = z$ .

### Example 4

Consider the following fractional integro-differential equation (Hasan *et al.*, 2013):

$$D^{(5/6)}y(t) = \frac{3}{91} \frac{t^{1/6}\Gamma(5/6)(-91 + 216t^2)}{\pi} + (5 - 2e)t + \int_0^1 te^\tau(\tau - \tau^3)d\tau, t \in [0,1], y(0) = 0 \quad (29)$$

with the exact solution as:

$$y(t) = t - t^3 \tag{30}$$

Comparison between analytical and ChNN solutions and plot of absolute error with the active function as,  $F(z) = z$  are presented in Fig. 12 and 13, respectively.

**Example 5**

Consider the following fractional integro-differential equation (Changqing and Jianhua, 2013):

$$D^{(3/4)}y(t) = \frac{6t^{9/4}}{\Gamma(13/4)} - \frac{t^2 e^t}{5} y(t) + \int_0^t e^t \tau y(\tau) d\tau, t \in [0,1], y(0) = 0 \tag{31}$$

with the exact solution as:

$$y(t) = t^3 \tag{32}$$

Comparison between analytical and ChNN solutions with active function as,  $F(z) = z$  is showed in Fig. 14. Plot of absolute error between analytical and ChNN solutions with  $F(z) = z$  is showed in Fig. 15.

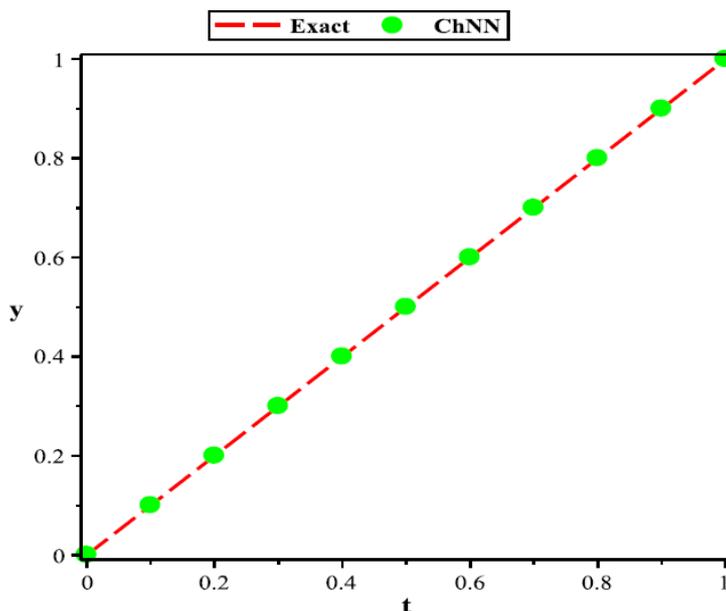


Fig. 8. Numerical results of the exact and ChNN solutions when  $F(z) = z$  (Example 2)

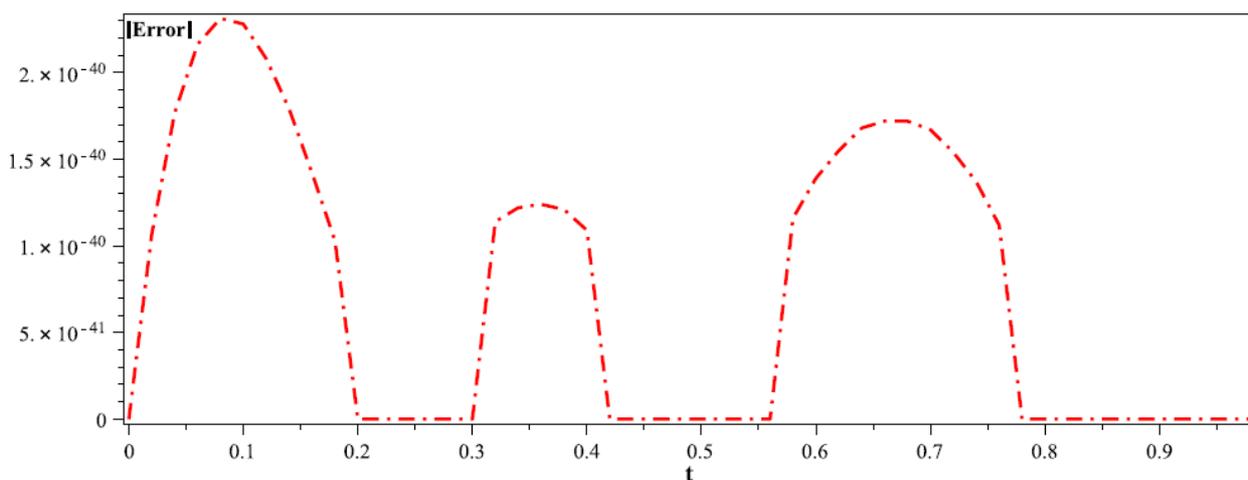


Fig. 9. Plot of absolute error between the exact and ChNN solutions when  $F(z) = z$  (Example 2)

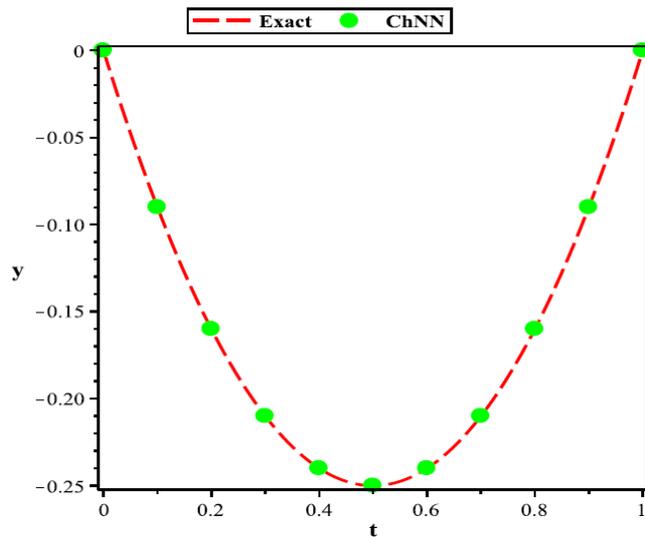


Fig. 10. Numerical results of the exact and ChNN solutions when  $F(z) = z$  (Example 3)

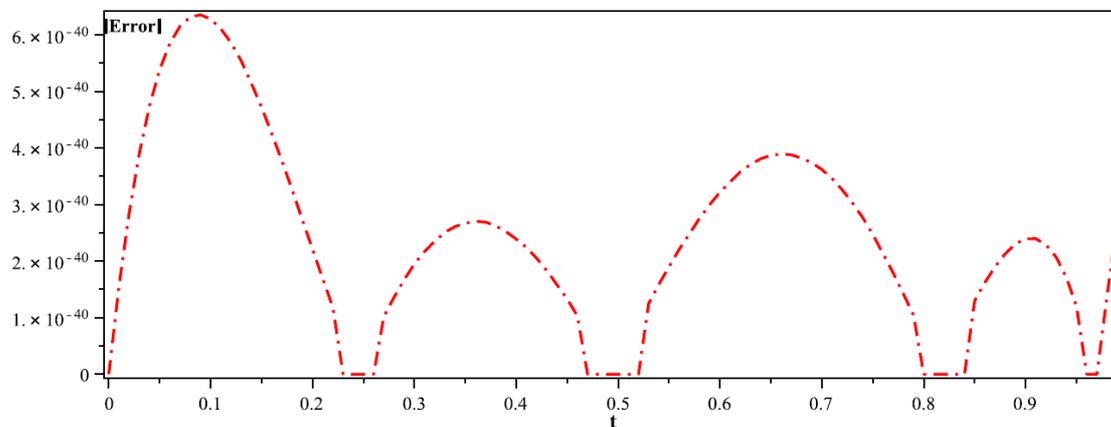


Fig. 11. Plot of absolute error between the exact and ChNN solutions when  $F(z) = z$  (Example 3)

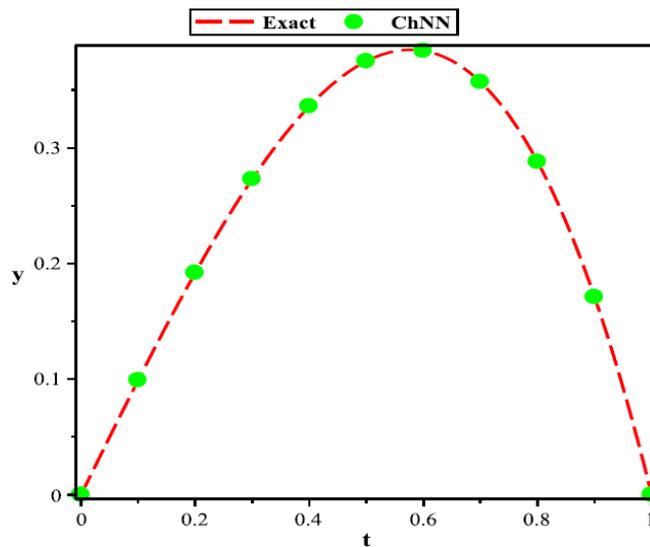


Fig. 12. Numerical results of the exact and ChNN solutions when  $F(z) = z$  (Example 4)

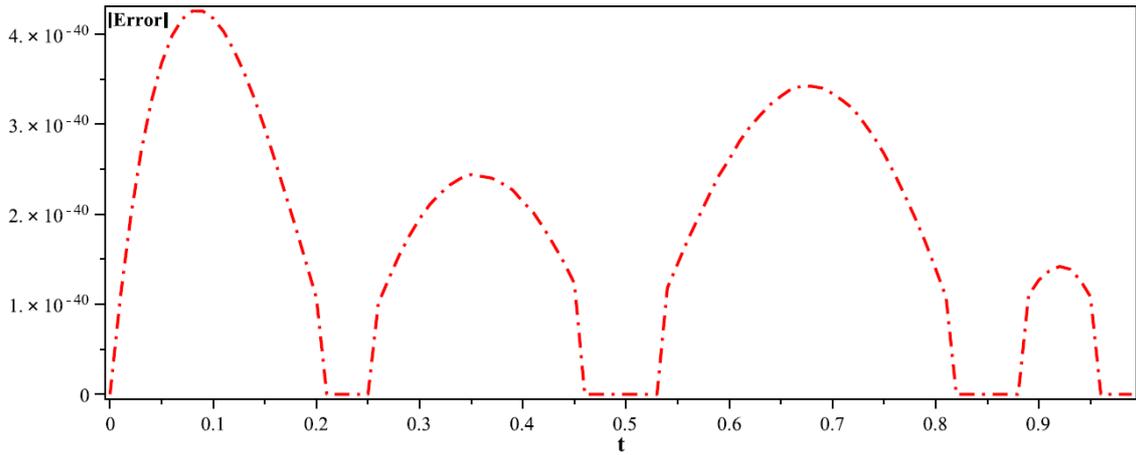


Fig. 13. Plot of absolute error between the exact and ChNN solutions when  $F(z) = z$  (Example 4)

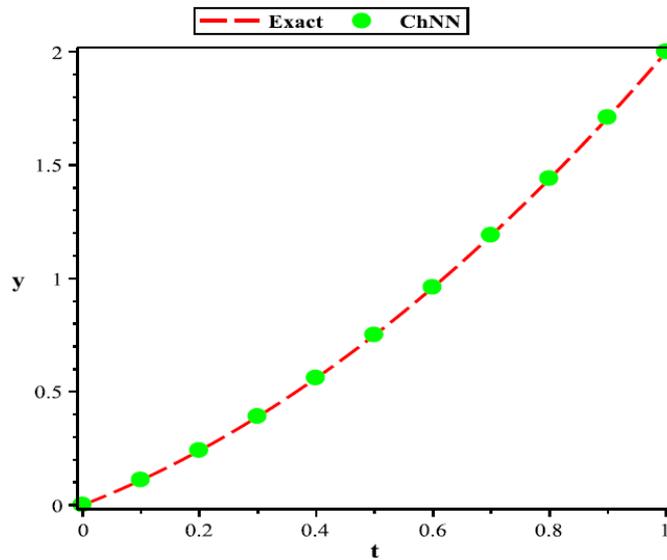


Fig. 14. Numerical results of the exact and ChNN solutions when  $F(z) = z$  (Example 5)

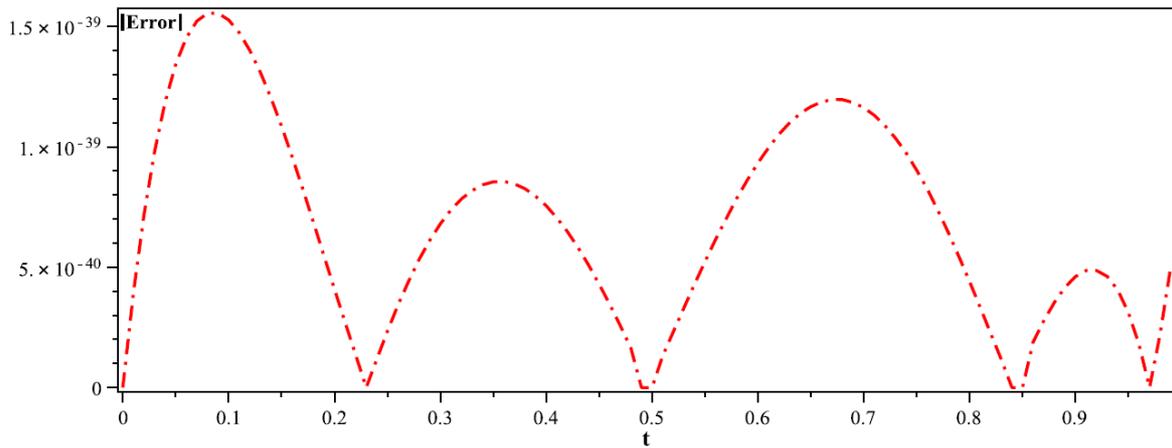


Fig. 15. Plot of the absolute error between the exact and ChNN solutions when  $F(z) = z$  (Example 5)

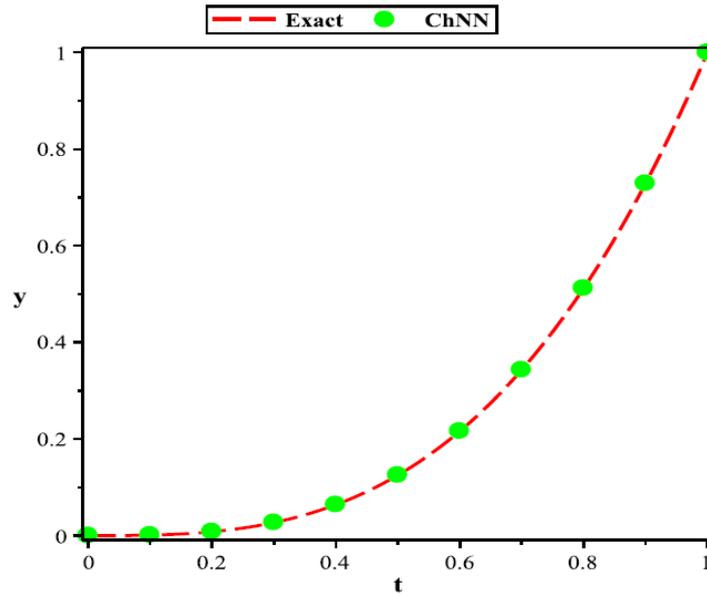


Fig. 16. Numerical results of Exact and ChNN solutions when  $F(z) = z$  (Example 6)

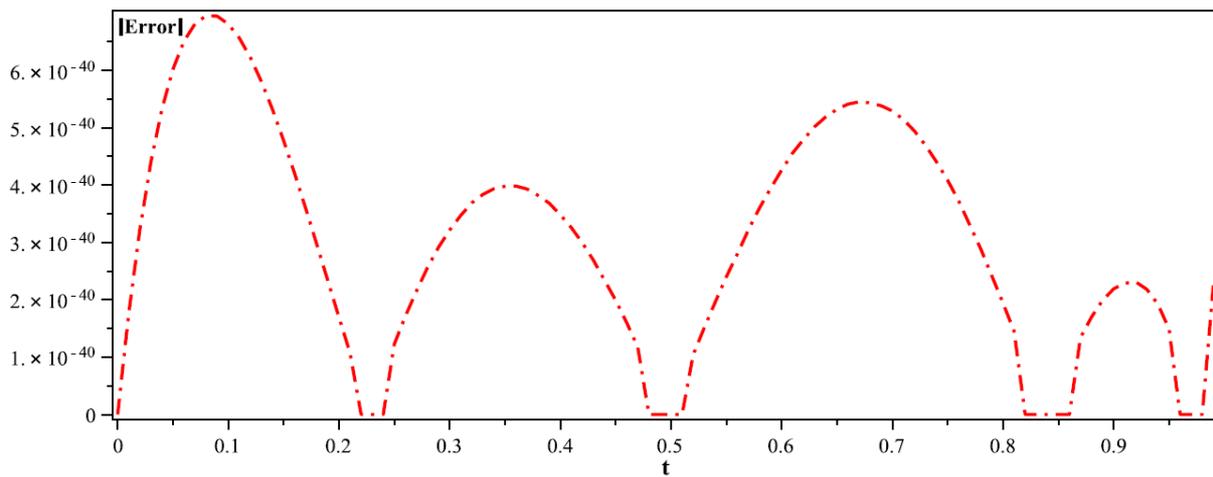


Fig. 17. Plot of absolute error between the exact and ChNN solutions when  $F(z) = z$  (Example 6)

**Example 6**

Consider the following fractional integro-differential equation (Mittal and Nigam, 2008):

$$D^{0.75}y(t) = \left[ -\frac{t^2 e^t}{5} \right] y(t) + \frac{6t^{2.25}}{\Gamma(3.25)} t^{1.5} \int_0^t e^t [\tau y(\tau)] d\tau \quad (33)$$

with the initial conditions  $y(0) = 0$  and exact solution as:

$$y(t) = t^3 \quad (34)$$

Figure 16 shows comparison between analytical and ChNN solutions with active function as,  $F(z) = z$ . Figure 17 presents plot of absolute error between analytical and ChNN solutions with  $F(z) = z$ .

**Conclusion**

Chebyshev neural network is applied for solving the linear and the nonlinear fractional integro-differential equations. A Chebyshev neural network with single layer is presented to prevail the difficulty and hardness of this type of equations. Numerical results from the proposed method are compared with

the analytical solutions. The maximum absolute error between exact and Chebyshev neural network solutions with  $F(z) = z$ , is showing a good agreement between Chebyshev neural network and analytical solutions. Comparisons of the obtained results from Chebyshev neural network with exact solutions show that this method is a capable tool for solving this kind of the linear and the nonlinear problems. This method can be applied to solve any kind of the complex ordinary and the partial differential equations from the fractional order.

### Acknowledgment

The author would like to thank the referees for their valuable comments that help to improve the manuscript.

### Author's Contributions

**Sarkhosh S. Chaharborj:** Analyzing, programing, producing the results and writing the paper.

**Shahriar S. Chaharborj:** Offering ideas, encouragement, proof read text and equations.

**Yaghoob Mahmoudi:** Supervising the contents and flow of the paper.

### Ethics

The authors confirm that they have read and approved the manuscript and no ethical issues involved.

### References

- Ahmed, S. and S.A.H. Salh, 2011. Generalized Taylor matrix method for solving linear integro-fractional differential equations of Volterra type. *Applied Math. Sci.*, 5: 1765-1780.
- Bhrawy, A.H. and A.S. Aloji, 2013. The operational matrix of fractional integration for shifted Chebyshev polynomials. *Applied Math. Lett.*, 26: 25-31. DOI: 10.1016/j.aml.2012.01.027
- Bhrawy, A.H. and M.A. Alghamdi, 2012. A shifted Jacobi-Gauss-Lobatto collocation method for solving nonlinear fractional Langevin equation involving two fractional orders in different intervals. *Boundary Value Prob.*, 2012: 62-62. DOI: 10.1186/1687-2770-2012-62
- Bianca, C., F. Pappalardo and S. Motta, 2009. The MWF method for kinetic equations system. *Comput. Math. Applic.*, 57: 831-840. DOI: 10.1016/j.camwa.2008.09.018
- Changqing, Y. and H. Jianhua, 2013. Numerical solution of integro-differential equations of fractional order by Laplace decomposition method. *Wseas Trans. Math.*, 12: 1173-1183.
- Doha, E.H., A.H. Bhrawy and S.S. Ezz-Eldien, 2011. Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. *Applied Math. Modell.*, 35: 5662-5672. DOI: 10.1016/j.apm.2011.05.011
- Fadi, A., E.A. Rawashdeh and H.M. Jaradat, 2011. Analytic solution of fractional integro-differential equations. *Annals Univ. Craiova Math. Comput. Sci.*, 38: 1-10.
- Grigorenko, I. and E. Grigorenko, 2003. Chaotic dynamics of the fractional Lorenz system. *Phys. Rev. Lett.*, 91: 034101-4. DOI: 10.1103/PhysRevLett.91.034101
- Hasan, B., B. HacıMehmet and F.B. BinMuhammad, 2013. The analytical solution of some fractional ordinary differential equations by the Sumudu transform method. *Abstract Applied Anal.*, 2013: 203875-203875. DOI: 10.1155/2013/203875
- Hoda, S.A. and H.A. Nagla, 2011. Neural network methods for mixed boundary value problems. *Int. J. Nonlinear Sci.*, 11: 312-316.
- Irandoost-Pakchin, S. and S. Abdi-Mazraeh, 2013. Exact solutions for some of the fractional integro-differential equations with the nonlocal boundary conditions by using the modification of He's variational iteration method. *Int. J. Adv. Math. Sci.*, 1: 139-144.
- Irandoost-pakchin, S., H. Kheiri and S. Abdi-Mazraeh, 2013. Chebyshev cardinal functions: An effective tool for solving nonlinear Volterra and Fredholm integro-differential equations of fractional order. *Iran. J. Sci. Technol. Trans. A*, 37: 53-62.
- Mall, S. and S. Chakraverty, 2014. Chebyshev Neural Network based model for solving Lane-Emden type equations. *Applied Math. Comput.*, 247: 100-114. DOI: 10.1016/j.amc.2014.08.085
- Mall, S. and S. Chakraverty, 2015. Numerical solution of nonlinear singular initial value problems of Emden-Fowler type using Chebyshev Neural Network method. *Neurocomputing*, 149: 975-982. DOI: 10.1016/j.neucom.2014.07.036
- Miller, K.S. and B. Ross, 1993. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. 1st Edn., John Wiley and Sons Inc., New York, ISBN-10: 0471588849, pp: 384.
- Mittal, R.C. and R. Nigam, 2008. Solution of fractional integro-differential equations by Adomian decomposition method. *Int. J. Applied Math. Mechan.*, 4: 87-94.
- Podlubny, I., 2002. Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calculus Applied Anal.*, 5: 367-38.

- Saeed, R.K. and H.M. Sdeq, 2010. Solving a system of linear fredholm fractional integro-differential equations using homotopy perturbation method. Austral. J. Basic Applied Sci., 4: 633-638.
- Saeedi, H. and F. Samimi, 2012. He's homotopy perturbation method for nonlinear fredholm integro-differential equations of fractional order. Int. J. Eng. Res. Applic., 2: 52-56.
- Saha Ray, S., 2009. Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method. Commun. Nonlinear Sci. Numerical Simulat., 14: 1295-1306.  
DOI: 10.1016/j.cnsns.2008.01.010
- Yang, Y., Y. Chen and Y. Huang, 2014. Spectral-collocation method for fractional Fredholm integro-differential equations. J. Korean Math. Society, 51: 203-224. DOI: 10.4134/JKMS.2014.51.1.203