

Approximate Series Solution of Nonlinear, Fractional Klein-Gordon Equations Using Fractional Reduced Differential Transform Method

¹Eman Abuteen, ²Asad Freihat, ²Mohammed Al-Smadi, ³Hammad Khalil and ³Rahmat Ali Khan

¹Department of Applied Science, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11942, Jordan

²Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan

³Department of Mathematics, University of Malakand, Chakadara Dir (L), Khyber Pakhtunkhwa, Pakistan

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Corresponding Author:
Mohammed Al-Smadi
Department of Applied
Science, Ajloun College, Al-
Balqa Applied University,
Ajloun 26816, Jordan
Email: mhm.smadi@yahoo.com

Abstract: This analysis proposes an analytical-numerical approach for providing solutions of a class of nonlinear fractional Klein-Gordon equation subjected to appropriate initial conditions in Caputo sense by using the Fractional Reduced Differential Transform Method (FRDTM). This technique provides the solutions very accurately and efficiently in convergent series formula with easily computable coefficients. The behavior of the approximate series solution for different values of fractional-order α is shown graphically. A comparative study is presented between the FRDTM and Implicit Runge-Kutta approach to illustrate the efficiency and reliability of the proposed technique. Our numerical investigations indicate that the FRDTM is simple, powerful mathematical tool and fully compatible with the complexity of such problems.

Keywords: Nonlinear Partial Differential Equation, Fractional Calculus, Series Solution, Fractional Reduced Differential Transform Method, Caputo Time-Fractional Derivative

Introduction

Fractional Partial Differential Equations (FPDEs) are widely used in interpretation and modeling of many of realism matters appear in applied mathematics and physics including fluid mechanics, electrical circuits, diffusion, damping laws, relaxation processes, mathematical biology (Klimek, 2005; Kilbas *et al.*, 2010; Baleanu *et al.*, 2009; Jumarie, 2009; Ortigueira, 2010; Mainardi, 2010). Fractional derivatives provide more accurate models of real-world problems than integer-order derivatives; they are actually found to be a powerful tool to describe certain physical problems. The topic of fractional calculus is a significantly important, useful branch of mathematics, plays a critical and serious role to describe a complex dynamical behavior in tremendous scope of application fields, helps to understand the nature of matter as well as simplified the controlling design without any loss of hereditary behaviors and explain even more complex structures.

Consider the following nonlinear Klein-Gordon equations of one-dimensional time fractional model:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + bG(u(x,t)) = f(x,t), t \geq 0 \quad (1)$$

with initial conditions:

$$u(x,0) = g_0(x), \frac{\partial}{\partial t} u(x,0) = g_1(x)$$

where, a and b are real constants, $f(x,t)$, $g_0(x)$ and $g_1(x)$ are known analytical functions, $G(u)$ is a nonlinear function, u is an unknown function of x and t to be determined. This model is derived from well-known Klein-Gordon Equations (KGEs) by replacing the time order derivative with fractional derivative of order α .

The KGEs are fundamental class of nonlinear evolution equations arising in classical relativistic and quantum mechanics. It got a lot of attention for studying solitons and condensed matter physics (Yusufoglu, 2008; Sweilam *et al.*, 2012). On the other hand, analytical-numerical studies of the solution for the FKGEs with the Caputo or Riemann-Liouville fractional derivative were

used to handle these problems (Golmankhaneh and Baleanu, 2011; Gepreel and Mohamed, 2013; Yang *et al.*, 2014; Marasi and Karimi, 2014; Khader *et al.*, 2014). As we know, there is no classical method to handle the nonlinear FPDEs and provide its explicit solution due to the complexities of fractional calculus involving these equations. For this reason, we need a reliable numerical approach to find the coefficients of the fractional series solutions of such equations. During the past few decades, many numerical-analytical methods were developed for handling the FPDEs and their system. For examples of these methods, we refer to the work in (Hesameddini and Fotros, 2012; Moaddy *et al.*, 2011; Abdulaziz *et al.*, 2008; Hashim *et al.*, 2009; Odibat and Momani, 2006; Khalil *et al.*, 2015a; 2015b; 2015c; El-Ajou *et al.*, 2015; Abu-Gdairi *et al.*, 2015; Freihat and Al-Smadi, 2013; Momani *et al.*, 2014; Al-Smadi *et al.*, 2013; 2015; 2016). On the other hand, many applications for different problems by using other numerical algorithms can be found in (Abu Arqub *et al.*, 2012; 2013; 2015; Abu Arqub and Al-Smadi, 2014; Moaddy *et al.*, 2015; Komashynska and Al-Smadi, 2014; Komashynska *et al.*, 2016).

In this analysis, we intend the application of FRDTM to provide numerical analytical solutions for a class of nonlinear partial differential equations included some well-known fractional Klein-Gordon equations. The FRDTM has several advantages for dealing directly with suggested equations; it needs a few iterations to get high accuracy, it is very simple for obtaining analytical-approximate solutions in rapidly convergent formulas, it allows better significantly information in providing continuous representation of these approximations and it has the ability for solving other problems appearing in several scientific fields.

This article is organized as follows: In the next section, necessary details and preliminaries about the fractional calculus theory are briefly provided. In section 3, the procedure of the RDTM is presented to construct and predict the series solution for fractional PDEs (1). In section 4, numeric results for certain types of FKGEs are given to verify the validity and performance of the present method. Finally, this article ends with some concluding remarks.

Mathematical Preliminaries

The basic preliminaries, concepts and notations of fractional integrals and derivatives in Caputo definition (Caputo, 1967) are introduced as follows. Here, we adopt the Caputo fractional derivative, which is a modification of Riemann-Liouville, because the initial conditions that defined during the formulation of the system are similar to those conventional conditions of integer order.

Definition 1

A real function $u(x, t)$, $x \in \mathbb{R}$, $t > 0$ is said to be in the space C_{μ} , $\mu \in \mathbb{R}$, if there exists a real number $q > \mu$ such

that $u(x, t) = t^q u_1(x, t)$, where $u_1(x, t) \in C(\mathbb{R} \times [0, \infty))$ and it is said to be in the space C_{μ}^m if $\frac{\partial^m}{\partial t^m} u(x, t) \in C_{\mu}$, $m \in \mathbb{N}$.

Definition 2

The Riemann-Liouville integral operator of order $\alpha \geq 0$ of a function $u(x, t) \in C_{\mu}$, $\mu \geq -1$ is defined as:

$$J_t^{\alpha} u(x, t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} u(x, \xi) d\xi, & \alpha > 0, 0 < \xi < t \\ u(x, t), & \alpha = 0 \end{cases} \quad (2)$$

Consequently, the operator J_t^{α} has the following properties: For $u(x, t) \in C_{\mu}$, $\mu \geq -1$, $\alpha, \beta \geq 0$, $c \in \mathbb{R}$ and $\gamma > -1$, one can get:

- $J_t^{\alpha} J_t^{\beta} u(x, t) = J_t^{\alpha+\beta} u(x, t) = J_t^{\beta} J_t^{\alpha} u(x, t)$
- $J_t^{\alpha} c = \frac{c}{\Gamma(\alpha+1)} t^{\alpha}$
- $J_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$

Now, we introduce a modified fractional differential operator D_t^{α} proposed by Caputo as follows:

$$D_t^{\alpha} f(x) = J_t^{m-\alpha} (x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\eta)^{m-\alpha-1} f^{(m)}(\eta) d\eta, t \geq 0$$

For $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x \geq t$ and $f(x) \in C_{-1}^m$.

Definition 3

For m to be the smallest integer that exceeds α , the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as:

$$D_t^{\alpha} u(x, t) = \frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = \begin{cases} J_t^{m-\alpha} \left(\frac{\partial^m u(x, t)}{\partial t^m} \right), & 0 \leq m-1 < \alpha < m \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \alpha = m \in \mathbb{N} \end{cases} \quad (3)$$

Theorem 1

If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $u(x, t) \in C_{\gamma}^m$ and $\gamma \geq -1$, then $D_t^{\alpha} J_t^{\alpha} u(x, t) = u(x, t)$ and $J_t^{\alpha} D_t^{\alpha} u(x, t) = u(x, t) - \sum_{k=0}^{m-1} \frac{\partial^k u(x, 0^+)}{\partial t^k} \frac{t^k}{k!}$, where $t > 0$.

For more details about FDEs, see (Millar and Ross, 1993; Podlubny, 1999; Samko *et al.*, 1993).

Description of the Method

Let $u(x,t)$ be a function of two variables such that $u(x,t) = f(x)g(t)$, then from the properties of the one-dimensional Differential Transform (DT) method, we have:

$$u(x,t) = \sum_{i=0}^{\infty} f(i)x^i \sum_{j=0}^{\infty} g(j)t^j = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} U(i,j)x^i t^j \quad (4)$$

where $U(i,j) = f(i)g(j)$ is called the spectrum of $u(x,t)$.

Next, we assume that $u(x,t)$ is continuously differentiable function with respect to space variable x and time t .

Lemma 1 (Srivastava *et al.*, 2013)

Let $u(x,t)$ be an analytic function, then the FRDT of u is given by:

$$U_k(x) = \frac{1}{\Gamma(k\alpha + 1)} \left[\frac{\partial^{k\alpha} u(x,t)}{\partial t^{k\alpha}} \right]_{t=t_0} \quad (5)$$

where α is a parameter describing the order of time-fractional derivative in Caputo sense.

The inverse transformed of U_k is defined by:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) (t-t_0)^{k\alpha} \quad (6)$$

From Equations 5 and 6, we have that:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[\frac{\partial^{k\alpha} u(x,t)}{\partial t^{k\alpha}} \right]_{t=t_0} (t-t_0)^{k\alpha} \quad (7)$$

In particular, for $t = 0$, Equation 7 reduces to:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[\frac{\partial^{k\alpha} u(x,t)}{\partial t^{k\alpha}} \right]_{t=t_0} t^{k\alpha} \quad (8)$$

Moreover, if $\alpha = 1$, then the FRDT of Equation 7 reduces to the classical RDT method.

From the above lemma, the fundamental operations of the FRDTM are given by the following theorems (Srivastava *et al.*, 2014):

Theorem 2

Let $u(x,t)$, $v(x,t)$ and $w(x,t)$ be any analytic functions such that $u(x,t) = R_D^{-1}[U_k(x)]$, $v(x,t) = R_D^{-1}[V_k(x)]$ and $w(x,t) = R_D^{-1}[W_k(x)]$, then:

- If $u(x,t) = v(x,t) \pm w(x,t)$, then $U_k(x) = V_k(x) \pm W_k(x)$
- If $u(x,t) = av(x,t)$, then $U_k(x) = aV_k(x)$, a is an arbitrary constant
- If $u(x,t) = x^m t^n v(x,t)$, then $U_k(x) = V_{k-n}(x)$
- If $u(x,t) = x^m t^n$, then $U_k(x) = x^m \delta(k-n)$, $\delta(k) = \begin{cases} 1, & k=0 \\ 0 & k \neq 0 \end{cases}$
- If $u(x,t) = v(x,t) w(x,t)$, then $U_k(x) = \sum_{r=0}^k V_r(x) W_{k-r}(x) = \sum_{r=0}^k W_r(x) V_{k-r}(x)$

Theorem 3

Let $u(x,t)$ and $v(x,t)$ be any two analytic functions such that $u(x,t) = R_D^{-1}[U_k(x)]$ and $v(x,t) = R_D^{-1}[V_k(x)]$, then:

- If $u(x,t) = \frac{\partial^r}{\partial x^r} v(x,t)$, then $U_k(x) = \frac{\partial^r}{\partial x^r} V_k(x)$
- If $u(x,t) = \frac{\partial^{r\alpha}}{\partial t^{r\alpha}} v(x,t)$, then $U_k(x) = \frac{\Gamma(\alpha k + \alpha r + 1)}{\Gamma(\alpha k + 1)} V_{k+r}(x)$

Corollary 1

$$\text{If } u(x,t) = e^{\lambda t + \mu x}, \text{ then } U_k(x) = \frac{\lambda^k}{k!} e^{\mu x}.$$

Corollary 2

$$\text{If } u(x,t) = \sin(\eta x + \omega t), v(x,t) = \cos(\eta x + \omega t), \text{ then } U_k(x) = \frac{\omega^k}{k!} \sin\left(\eta x + \frac{\pi k}{2}\right) \text{ and } V_k(x) = \frac{\omega^k}{k!} \cos\left(\eta x + \frac{\pi k}{2}\right).$$

The reader is referred to (Keskin and Oturanc, 2009; Abazari and Abazari, 2012; Secer, 2012; Sohail and Mohyud-Din, 2012; Al-Amr, 2014) and the references therein to know more details about the reduced differential transform technique, including their applications in various kinds of differential equations.

Now, by applying the FRDTM to Equation 1, we obtain the following recurrence relation formula:

$$\frac{\Gamma(\alpha k + 2\alpha + 1)}{\Gamma(\alpha k + 1)} U_{k+2}(x) = \frac{\partial^2}{\partial x^2} U_k(x) - aU_k(x) - bG[U_k(x)] + F_k(x) \quad (9)$$

where $F_k(x)$ and $G[U_k(x)]$ are the reduced transformation of the functions $f(x,t)$ and $G(u(x,t))$, respectively.

Using the initial conditions, we have:

$$U_0(x) = g_0(x), U_1(x) = g_1(x) \quad (10)$$

Substituting Equation 10 into Equation 9 and by straightforward iterative calculation, we obtain the

values $U_k(x)$, for $k = 1, 2, 3, \dots$. Thus, inverse RDT of $\{U_k(x)\}_{k=1}^n$ yields that:

$$\tilde{u}_n(x, t) = \sum_{k=0}^n U_k(x) (t - t_0)^{k\alpha} \quad (11)$$

Therefore, the closed form solution given by:

$$u(x, t) = \lim_{x \rightarrow \infty} \tilde{u}_n(x, t) \quad (12)$$

Numerical Examples

In this section, some numerical examples are given to verify the simplicity and applicability of the present technique in finding approximate series solution for fractional KGEs. The simulation results indicate that the FRDT method is highly accurate and fully compatible with the complexity of the PDEs of fractional-order.

Example 4.1

We consider the following one-dimensional linear fractional Klein-Gordon equation:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) = 0, 0 < \alpha \leq 1 \quad (13)$$

with initial condition:

$$u(x, 0) = 1 + \sin(x)$$

where $x, t \geq 0$.

Applying the transformation (Li and He, 2010), we have:

$$\frac{\partial u(x, t)}{\partial T} - \frac{\partial^2 u(x, t)}{\partial x^2} = u(x, t)$$

By using the FRDTM of Equation 13, we have the recurrence relation formula:

$$U_{k+1}(x) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\frac{\partial^2}{\partial x^2} U_k(x) + U_k(x) \right] \quad (14)$$

with transformed initial data:

$$U_0(x) = 1 + \sin(x) \quad (15)$$

Substituting the condition (15) into Equation 14, we get the values successively $U_k(x)$, $k = 1, 2, 3, \dots$, as follows:

$$U_1(x) = \frac{1}{\Gamma(\alpha + 1)}, U_2(x) = \frac{1}{\Gamma(2\alpha + 1)}, \\ U_3(x) = \frac{1}{\Gamma(3\alpha + 1)}, \dots, U_k(x) = \frac{1}{\Gamma(k\alpha + 1)}$$

Thus, the approximate solution can be obtained by:

$$u(x, t) = 1 + \sin(x) + \frac{1}{\Gamma(\alpha + 1)} T \\ + \frac{1}{\Gamma(2\alpha + 1)} T^2 + \frac{1}{\Gamma(3\alpha + 1)} T^3 + \dots$$

The inverse RDTM is given by:

$$u(x, t) = 1 + \sin(x) + \frac{1}{\Gamma(\alpha + 1)} t^\alpha \\ + \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha} + \frac{1}{\Gamma(3\alpha + 1)} t^{3\alpha} + \dots \\ = \sin(x) + \sum_{k=0}^{\infty} \frac{1}{\Gamma(2\alpha + 1)} t^{k\alpha}$$

Consequently, the reduced inverse transformed of $U_k(x)$ follows the closed form solution. Setting $\alpha = 1$, the exact solution is:

$$u(x, t) = \sin(x) + e^t$$

To demonstrate the efficiency of the present method, we compare the FRDT approximation with the Implicit Runge-Kutta (IRK) method for $\alpha = 1$. Figure 1 shows the phase portrait of solutions for Example 4.1 using the FRDTM and IRKM for $t \in [0, 0.8]$ and $x \in [0, 4]$. The numerical results for different time levels of α are presented in Fig. 2. Here, we note that the approximate FRDT are efficiency for time-fractional KGE and very closed to the IRK solutions.

Example 4.2

We consider the nonlinear Klein-Gordon fractional model in the form:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^2 u(x, t)}{\partial x^2} + u^2(x, t) = 0, 0 < \alpha \leq 1 \quad (16)$$

with initial condition:

$$u(x, 0) = 1 + \sin(x)$$

Applying the transformation (Li and He, 2010), we have:

$$\frac{\partial u(x,t)}{\partial T} - \frac{\partial^2 u(x,t)}{\partial x^2} = u^2(x,t)$$

By using the FRDTM, we have the recurrence relation formula:

$$U_{k+1}(x) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1)+1)} \left[\frac{\partial^2}{\partial x^2} U_k(x) + \sum_{r=0}^k U_r(x) U_{k-r}(x) \right] \quad (17)$$

With RDTM of initial condition:

$$U_0(x) = 1 + \sin(x)$$

Following recurrence relation (17), the sequences components $U_k(x)$, $k = 1, 2, 3, \dots$, were computed using the Mathematica package, can be successively given by:

$$U_1(x) = \frac{-\Gamma(\alpha)}{\Gamma(2\alpha)} [1 + 3\sin(x) + \sin^2(x)],$$

$$U_2(x) = \frac{-\Gamma(\alpha)}{2\Gamma(3\alpha)} [12\cos(2x) - 25\sin(x) + \sin(3x) - 12]$$

$$U_3(x) = \frac{-\Gamma(\alpha)}{8(\Gamma(2\alpha))^2 \Gamma(4\alpha)} \left[\begin{array}{l} 2\Gamma(\alpha)\Gamma(3\alpha) \\ (-3 + \cos(2x) - 6\sin^2(x)) + 4(\Gamma(2\alpha))^2 \\ 49 - 98\cos(2x) \\ + \cos(4x) + 111\sin(x) - 23\sin(3x) \end{array} \right]$$

∴
and so on

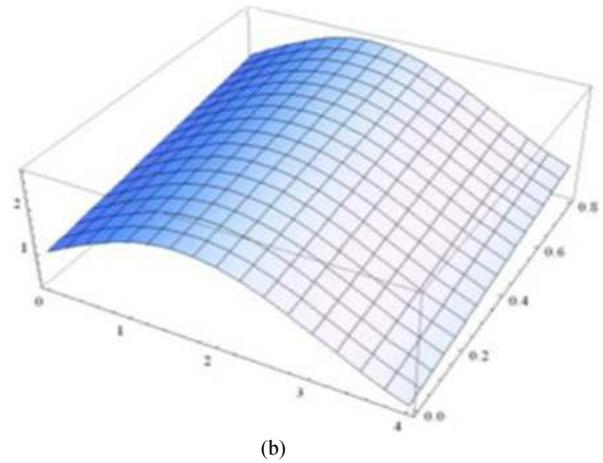
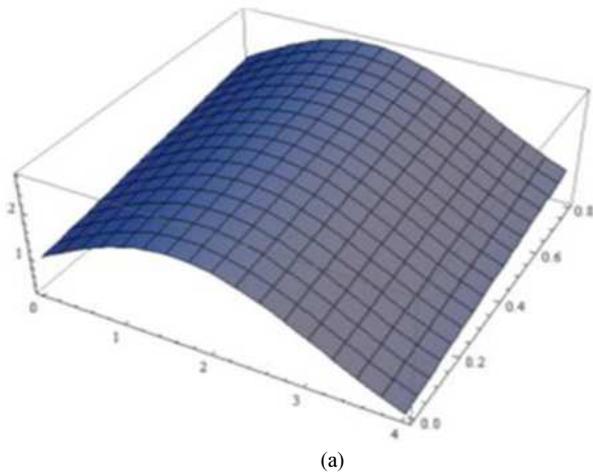


Fig. 1. Comparison of phase plot for $u(x,t)$ of Example 4.1 at $\alpha = 1$, $x \in [0,4]$ and $t \in [0,0.8]$: (a) the FRDTM; (b) implicit Runge-Kutta method

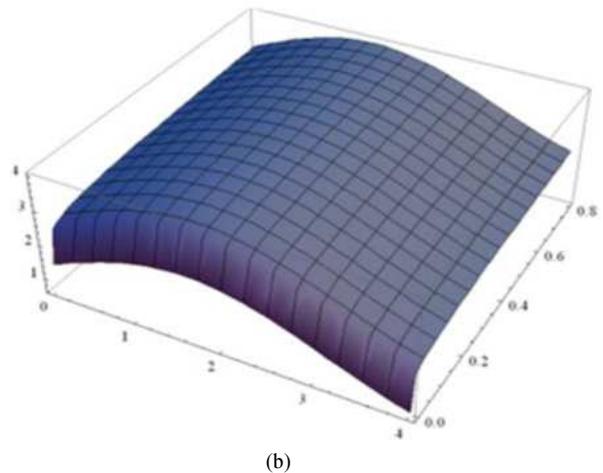
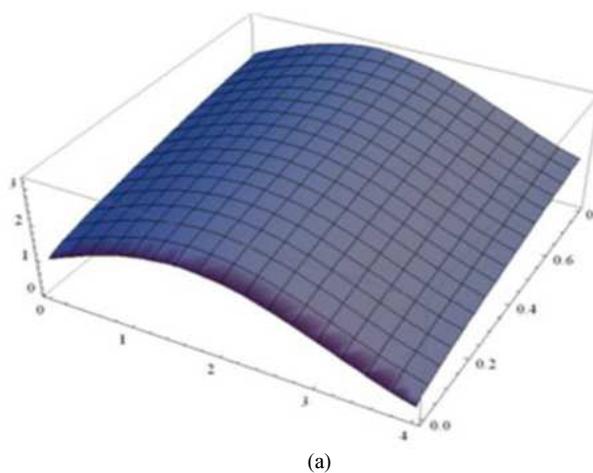


Fig. 2. Phase plot of the solution $u(x,t)$ of Example 4.1 using the FRDTM for $x \in [0,4]$ and $t \in [0,0.8]$: (a) at $\alpha = 0.7$; (b) at $\alpha = 0.1$

The approximate form solution is given by:

$$u(x,t) = 1 + \sin(x) + U_1(x)T + U_2(x)T^2 + U_3(x)T^3 + \dots$$

Accordingly, the inverse RDT is given by:

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} \\ &= 1 + \sin(x) - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} [1 + 3\sin(x) + \sin^2(x)]t^\alpha \\ &\quad - \frac{\Gamma(\alpha)}{2\Gamma(3\alpha)} [12\cos(2x) - 25\sin(x) + \sin(3x) - 12]t^{2\alpha} \\ &\quad - \frac{\Gamma(\alpha)}{8(\Gamma(2\alpha))^2\Gamma(4\alpha)} \left[\begin{array}{l} 2\Gamma(\alpha)\Gamma(3\alpha) \begin{pmatrix} -3 + \cos(2x) \\ -6\sin^2(x) \end{pmatrix} \\ +4(\Gamma(2\alpha))^2 \begin{pmatrix} 49 - 98\cos(2x) + \cos(4x) \\ +111\sin(x) - 23\sin(3x) \end{pmatrix} \end{array} \right] t^{3\alpha} \\ &\quad + \dots \end{aligned}$$

The series solution of u follows closed form solution. The geometric behaviors of the solution for Example 4.2 are studied by drawing the 3-dimensional space

figures of the FRDT approximate solution together with its corresponding IRK solution. Figure 3 shows the comparison between the FRDT approximate solution and IRK solution at $\alpha = 1$ for $x \in [-2,2]$ and $t \in [0,0.01]$. Whileas, Fig. 4 shows the solution behavior of the nonlinear FKGE for different specific cases of α in the domain $x \in [-2,2]$ and $t \in [0,0.01]$. The performance errors analysis are obtained by the FRDTM at $x = 2$ and summarized in Table 1. Numerically, it is showed that the RDT method is effective and accurate.

Example 4.3

We consider the nonlinear Klein-Gordon fractional model in the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) - u^3(x,t) = 0, 0 < \alpha \leq 1 \quad (18)$$

with initial condition:

$$u(x,0) = -\operatorname{sech}(x)$$

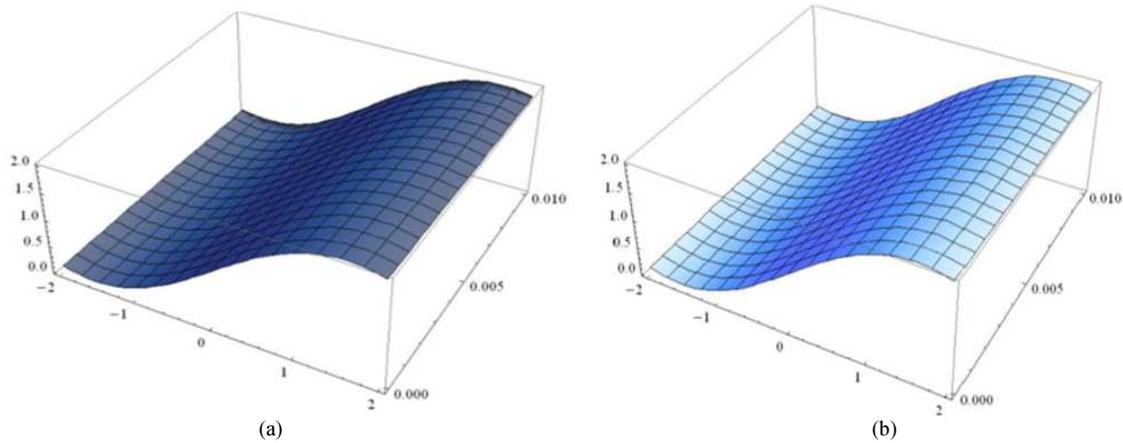


Fig. 3. Comparison of phase plot for $u(x,t)$ of Example 4.2 at $\alpha = 1$, $x \in [-2,2]$ and $t \in [0,0.01]$: (a) the FRDTM; (b) implicit RKM

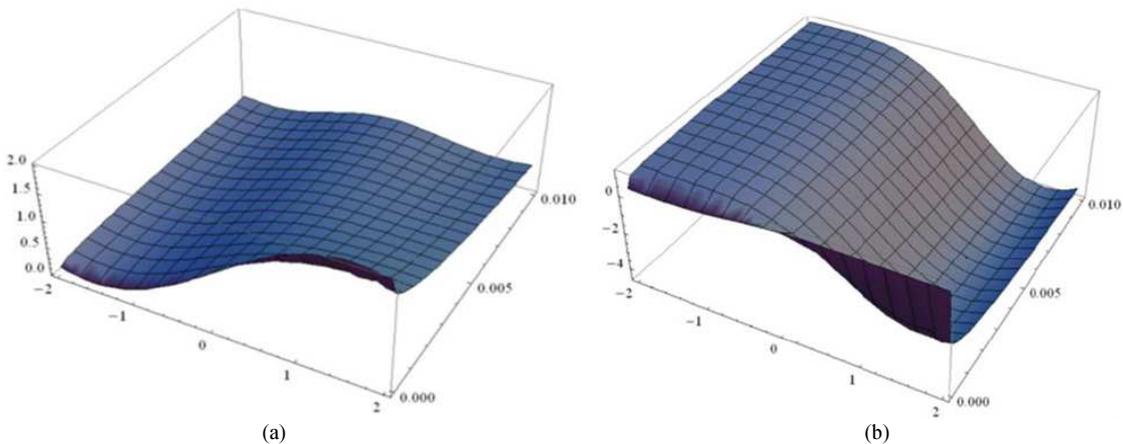


Fig. 4. Phase plot of the solution $u(x,t)$ of Example 4.2 using the FRDTM for $x \in [-2,2]$ and $t \in [0,0.01]$: (a) at $\alpha = 0.4$; (b) at $\alpha = 0.1$

Table 1. The error analysis for Example 4.2 when $\alpha = 1$ with $x = 2$

t	RDTM	IRKM	Absolute error	Relative error
0.000	1.909297426825682	1.909297426825682	0.00000	0.00000
0.001	1.904742712734773	1.904743294464910	5.81730×10^{-7}	3.05411×10^{-7}
0.002	1.900187998643864	1.900189162104138	1.16346×10^{-6}	6.12287×10^{-7}
0.003	1.895633284552955	1.895635029743366	1.74519×10^{-6}	9.20636×10^{-7}
0.004	1.891078570462046	1.891080897382594	2.32692×10^{-6}	1.23047×10^{-6}
0.005	1.886523856371138	1.886526765021823	2.90865×10^{-6}	1.54180×10^{-6}
0.006	1.881969142280229	1.881972632661051	3.49038×10^{-6}	1.85464×10^{-6}
0.007	1.877414428189319	1.877418500300279	4.07211×10^{-6}	2.16899×10^{-6}
0.008	1.872859714098411	1.872864367939507	4.65384×10^{-6}	2.48488×10^{-6}
0.009	1.868305000007502	1.868310235578735	5.23557×10^{-6}	2.80230×10^{-6}
0.010	1.863750285916593	1.863756103217964	5.81730×10^{-6}	3.12128×10^{-6}

Applying the transformation (Li and He, 2010), we have:

$$\frac{\partial u(x,t)}{\partial T} = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + u^3(x,t)$$

By using the FRDTM, we have the recurrence relation formula:

$$U_{k+1}(x) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\frac{\partial^2}{\partial x^2} U_k(x) - U_k(x) + \sum_{r=0}^k \sum_{l=0}^r U_l(x) U_{k-r-l}(x) \right] \quad (19)$$

with RDT of initial condition:

$$U_0(x) = -\operatorname{sech}(x)$$

Following recurrence relation (19), the sequences components $U_k(x)$, $k = 1, 2, 3, \dots$, were computed using the Mathematica 9.0 package, can be successively given by:

$$U_1(x) = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \left[(\operatorname{sech}(x))^3 \right],$$

$$U_2(x) = \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \left[(\operatorname{sech}(x))^5 (4 \cosh(2x) - 5) \right],$$

$$U_3(x) = \frac{\Gamma(\alpha)}{(\Gamma(2\alpha))^2 \Gamma(4\alpha)} \left[(\operatorname{sech}(x))^7 \left(\begin{matrix} 123 - 112 \cosh(2x) \\ + 8 \cosh(4x) \\ (\Gamma(2\alpha))^2 3\Gamma(\alpha)\Gamma(3\alpha) \end{matrix} \right) \right],$$

\vdots
and soon

By taking the inverse RDT of $\{U_k(x)\}_{k=0}^n$, we have:

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} \\ &= -\operatorname{sech}(x) + \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (\operatorname{sech}(x))^3 t^\alpha \\ &\quad + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \left[(\operatorname{sech}(x))^5 (4 \cosh(2x) - 5) \right] t^{2\alpha} \\ &\quad + \frac{\Gamma(\alpha)}{(\Gamma(2\alpha))^2 \Gamma(4\alpha)} \left[(\operatorname{sech}(x))^7 \left(\begin{matrix} 123 - 112 \cosh(2x) \\ + 8 \cosh(4x) \\ (\Gamma(2\alpha))^2 - 3(\alpha)\Gamma(3\alpha) \end{matrix} \right) \right] t^{3\alpha} + \dots \end{aligned}$$

Figure 5 and 6 show the solution behavior for the nonlinear time-fractional KGE in Example 4.3 by using the RDTM and IRKM at $x \in [-2, 2]$ and $t \in [0, 0.01]$ for different particular values of α .

Example 4.4

We consider the nonlinear Klein-Gordon fractional model in the form:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) = 0, 1 < \alpha \leq 2 \quad (20)$$

with initial conditions:

$$u(x,0) = 1 + \sin(x), \quad \frac{\partial}{\partial t} u(x,0) = 0 \quad (21)$$

Using the transformation (Li and He, 2010), we have:

$$\frac{\partial u(x,t)}{\partial T} = \frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t)$$

By using the FRDTM, we obtain the recurrence relations formula as:

$$U_{k+2}(x) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+2) + 1)} \left[\frac{\partial^2}{\partial x^2} U_k(x) - \sum_{r=0}^k U_r(x) U_{k-r}(x) \right] \quad (22)$$

with RDT of initial condition:

$$U_0(x) = 1 + \sin(x), U_1(x) = 0$$

Following recurrence relation (22) and by straightforward iterative steps, yields:

$$\begin{aligned} U_2(x) &= -\frac{\Gamma(\alpha)}{\Gamma(3\alpha)} [1 + 3\sin(x) + \sin^2(x)], U_3(x) = 0, \\ U_4(x) &= -\frac{\Gamma(\alpha)}{2\Gamma(4\alpha)} [12\cos(2x) - 25\sin(x) + \sin(3x) - 12], U_5(x) = 0, \\ U_6(x) &= -\frac{\Gamma(\alpha)}{8(\Gamma(2\alpha))^2 \Gamma(5\alpha)} \left[\begin{array}{l} 2\Gamma(\alpha)\Gamma(3\alpha) \begin{pmatrix} -3 + \cos 2x \\ -6\sin^2 x \end{pmatrix} \\ + 4(\Gamma(2\alpha))^2 \begin{pmatrix} -12 + 12\cos 2x \\ -25\sin x + \sin 3x \end{pmatrix} \end{array} \right], \\ U_7(x) &= 0 \\ &\vdots \\ &\text{and so on} \end{aligned}$$

Table 2. Numerical results of u when $x = 2$ for Example 4.4

t	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
0.00	1.9092974268257	1.9092974268257	1.9092974268257	1.9092974268257
0.02	1.8577557126584	1.8953639720340	1.9056740630824	1.9083864840075
0.04	1.7867098804114	1.8698876653513	1.8971099324942	1.9056536555530
0.06	1.7057992950300	1.8368970719530	1.8845189771575	1.9010989414620
0.08	1.6177334620125	1.7978297884919	1.8683037456485	1.8947223417348
0.10	1.5239332295153	1.7535166664471	1.8487201924577	1.8865238563711
0.12	1.4252945743833	1.7045182992625	1.8259529888198	1.8765034853711
0.14	1.3224416173263	1.6512462328457	1.8001449727689	1.8646612287348
0.16	1.2158375439572	1.5940193350310	1.7714116686258	1.8509970864620
0.18	1.1058416180037	1.5330941474492	1.7398494741889	1.8355110585530
0.20	0.9927417362082	1.4686828986574	1.7055407099216	1.8182031450075

Table 3. The numerical results for Example 4.4 when $\alpha = 2$ with $x = 2$.

t	FRDTM	IRKM	Absolute error	Relative error
0.00	1.90929742682568	1.90929742682568	0.00000	0.00000
0.02	1.90838648400750	1.90838660035353	1.16346×10^{-7}	6.09656×10^{-8}
0.04	1.90565365555296	1.90565412093706	4.65384×10^{-6}	2.44212×10^{-7}
0.06	1.90109894146205	1.90109998857629	1.04711×10^{-6}	5.50794×10^{-7}
0.08	1.89472234173477	1.89472420327121	1.86154×10^{-6}	9.82484×10^{-7}
0.10	1.88652385637114	1.88652676502182	2.90865×10^{-6}	1.54180×10^{-6}
0.12	1.87650348537114	1.87650767382812	4.18846×10^{-6}	2.23205×10^{-6}
0.14	1.86466122873478	1.86466692969012	5.70096×10^{-6}	3.05736×10^{-6}
0.16	1.85099708646205	1.85100453260780	7.44615×10^{-6}	4.02276×10^{-6}
0.18	1.83551105855296	1.83552048258118	9.42403×10^{-6}	5.13425×10^{-6}
0.20	1.81820314500750	1.81821477961024	1.16346×10^{-5}	6.39892×10^{-6}

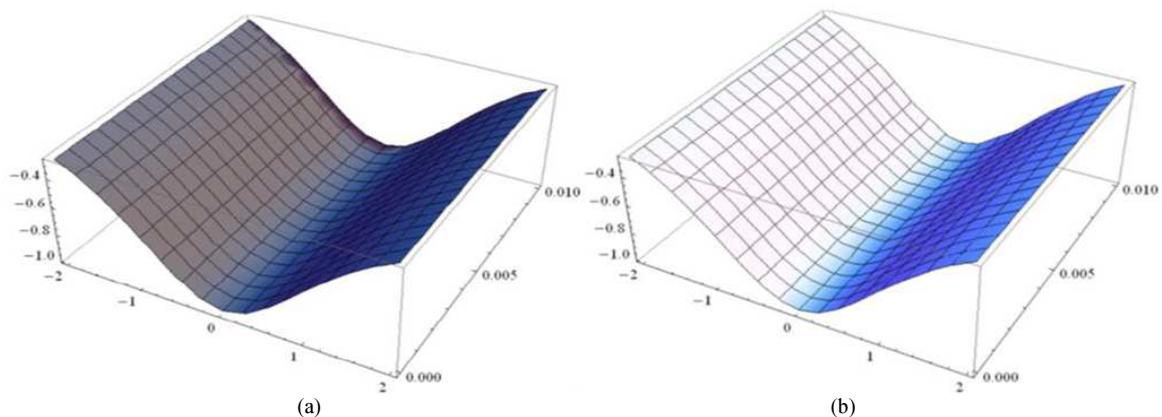


Fig. 5. Comparison of phase plot for $u(x,t)$ of Example 4.3 at $\alpha = 1$, $x \in [-2,2]$ and $t \in [0,0.01]$: (a) the FRDTM; (b) implicit Runge-Kutta method

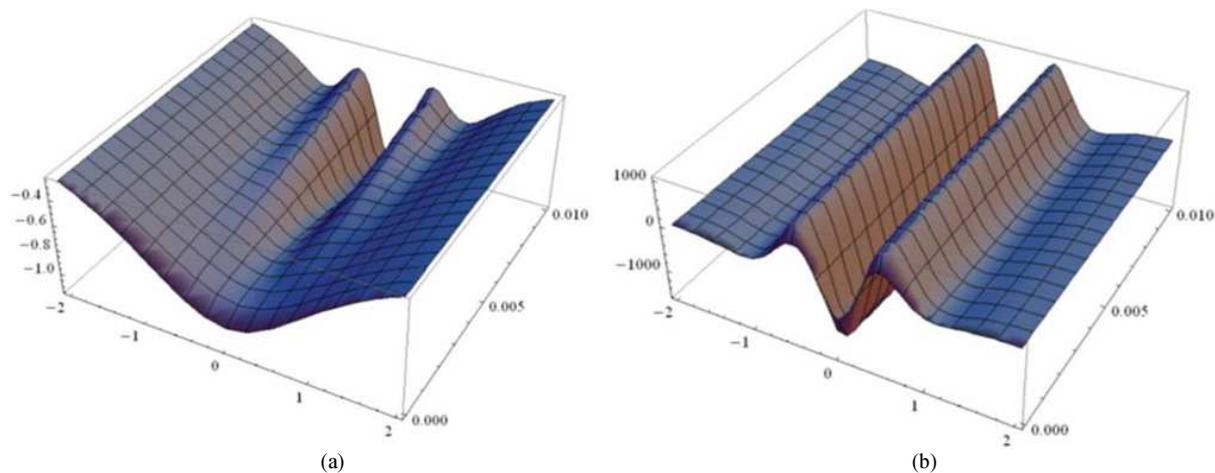


Fig. 6. Phase plot of the solution $u(x,t)$ of Example 4.2 using the FRDTM for $x \in [-2,2]$ and $t \in [0,0.01]$: (a) at $\alpha = 0.4$; (b) at $\alpha = 0.1$

The approximate form solution is obtained by:

$$u(x,t) = 1 + \sin(x) + U_1(x)T + U_2(x)T^2 + U_3(x)T^3 + U_4(x)T^4 + \dots$$

Consequently, the reduced inverse DT of $U_k(x)$ will be given by:

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} \\ &= 1 + \sin(x) - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} [1 + 3\sin(x) + \sin^2(x)]t^{2\alpha} \\ &\quad - \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} [1 + 3\sin(x) + \sin^2(x)]t^{3\alpha} \\ &\quad - \frac{\Gamma(\alpha)}{2\Gamma(4\alpha)} [12\cos(2x) - 25\sin(x) + \sin(3x) - 12]t^{4\alpha} \\ &\quad - \frac{\Gamma(\alpha)}{8(\Gamma(2\alpha))^2\Gamma(5\alpha)} \left[\begin{matrix} 2\Gamma(\alpha)\Gamma(3\alpha) \begin{pmatrix} -3 + \cos 2x \\ -6\sin^2 x \end{pmatrix} \\ + 4(\Gamma(2\alpha))^2 \begin{pmatrix} -12 + 12\cos 2x \\ -25\sin x + \sin 3x \end{pmatrix} \end{matrix} \right] t^{4\alpha} + \dots \end{aligned}$$

In Table 2, we summarized numerical values for the solution of Equation 20 when $\alpha = 1.25, 1.5, 1.75$ and $\alpha = 2$ with $x = 2$. Also numerical results using our approximation and the implicit Runge-Kutta method for $\alpha = 2$ are summarized in Table 3. From these tables, it is easier to observe that the numerical approximations are in agreement with each other's and with the IRK method. Also, they have same behavior as those obtained using the implicit Runge-Kutta method.

Concluding Remarks

In this study, we present numerical algorithm for finding approximate form solutions of a class of Klein-Gordon fractional model based upon FRDTM. This method was used directly without employing linearization and perturbation. The efficiency and capability of the present algorithm have been checked via several illustrated examples. The results reveal the complete reliability of this method with a great potential in scientific applications. Finally, we conclude that the FRDTM is very powerful, straightforward and effective to obtain analytical numerical solutions of a wide variety problems related to fractional PDEs applied in mathematics, physics and engineering. Computations of this paper have been carried out by using the computer package of Mathematica 9.

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Author's Contributions

All authors have contributed equally.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

Al-Amr, M.O., 2014. New applications of reduced differential transform method. *Alex. Eng. J.*, 53: 243-247. DOI: 10.1016/j.aej.2014.01.003

- Abazari, R. and M. Abazari, 2012. Numerical simulation of generalized Hirota-Satsuma coupled KdV equation by RDTM and comparison with DTM. *Comm. Nonlin. Sci. Num. Simul.*, 17: 619-629. DOI: 10.1016/j.cnsns.2011.05.022
- Abdulaziz, O., I. Hashim and A. Saif, 2008. Series solutions of time-fractional PDEs by homotopy analysis method. *Differential Equat. Nonlinear Mechan.*, 2008: 686512-686527. DOI: 10.1155/2008/686512
- Abu Arqub, O., M. Al-Smadi and S. Momani, 2012. Application of reproducing kernel method for solving nonlinear Fredholm-Volterra integro-differential equations. *Abstract Applied Anal.*, 2012: 839836-839851. DOI: 10.1155/2012/839836
- Abu Arqub, O., M. Al-Smadi and N. Shawagfeh, 2013. Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method. *Applied Math. and Comput.*, 219: 8938-8948. DOI: 10.1016/j.amc.2013.03.006
- Abu Arqub, O. and M. Al-Smadi, 2014. Numerical algorithm for solving two-point, second order periodic boundary value problems for mixed integro-differential equations. *Applied Math. Comput.*, 243: 911-922. DOI: 10.1016/j.amc.2014.06.063
- Abu Arqub, O., M. Al-Smadi, S. Momani and T. Hayat, 2015. Numerical solutions of fuzzy differential equations using reproducing kernel hilbert space method. *Soft Comput.* DOI: 10.1007/s00500-015-1707-4
- Abu-Gdairi, R., M. Al-Smadi and G. Gumah, 2015. An expansion iterative technique for handling fractional differential equations using fractional power series scheme. *J. Math. Stat.*, 11: 29-38. DOI: 10.3844/jmssp.2015.29.38
- Al-Smadi, M., A. Freihat, M. Abu Hammad, S. Momani and O. Abu Arqub, 2016. Analytical approximations of partial differential equations of fractional order with multistep approach. *J. Comput. Theoretical Nanosci.*
- Al-Smadi, M., A. Freihat, O. Abu Arqub and N. Shawagfeh, 2015. A novel multistep generalized differential transform method for solving fractional-order Lu chaotic and hyperchaotic systems. *J. Comput. Anal. Applic.*, 19: 713-724.
- Al-Smadi, M., O. Abu Arqub and S. Momani, 2013. A computational method for two-point boundary value problems of fourth-order mixed integro-differential equations. *Math. Problems Eng.*, 2013: 832074-832083. DOI: 10.1155/2013/832074
- Baleanu, D., A.K. Golmankhaneh and M.C. Baleanu, 2009. Fractional electromagnetic equations using fractional forms. *Int. J. Theor. Phy.*, 48: 3114-3123. DOI: 10.1007/s10773-009-0109-8
- Caputo, M., 1967. Linear models of dissipation whose Q is almost frequency independent-II. *Geophys. J. Int.*, 13: 529-539. DOI: 10.1111/j.1365-246X.1967.tb02303.x
- El-Ajou, A., O. Abu Arqub and M. Al-Smadi, 2015. A general form of the generalized Taylor's formula with some applications. *Applied Math. Comp.*, 256: 851-859. DOI: 10.1016/j.amc.2015.01.034
- Freihat, A. and M. Al-Smadi, 2013. A new reliable algorithm using the generalized differential transform method for the numeric-analytic solution of fractional-order Liu chaotic and hyperchaotic systems. *Pensee J.*, 75: 263-276.
- Gepreel, K.A. and M.S. Mohamed, 2013. Analytical approximate solution for nonlinear space-time fractional Klein-Gordon equation. *Chin. Phys. B.* DOI: 10.1088/1674-1056/22/1/010201
- Golmankhaneh, A.K. and D. Baleanu, 2011. On nonlinear fractional Klein-Gordon equation. *Sign. Proces.*, 91: 446-451. DOI: 10.1016/j.sigpro.2010.04.016
- Hashim, I., O. Abdulaziz and S. Momani, 2009. Homotopy analysis method for fractional IVPs. *Commun. Nonlin. Sci. Numerical Simulat.*, 14: 674-684. DOI: 10.1016/j.cnsns.2007.09.014
- Hesameddini, E. and F. Fotros, 2012. Solution for time-fractional coupled Klein-Gordon Schrodinger equation using decomposition method. *Int. Math. Foru.*, 7: 1047-1056.
- Jumarie, G., 2009. Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. *Applied Math. Lett.*, 22: 378-385. DOI: 10.1016/j.aml.2008.06.003
- Keskin, Y. and G. Oturanc, 2009. Reduced differential transform method for partial differential equations. *Int. J. Nonlin. Sci. Num. Simul.*, 10: 741-749. DOI: 10.1515/IJNSNS.2009.10.6.741
- Kilbas, A.A., H.M. Srivastava and J.J. Trujillo, 2010. *Theory and Applications of Fractional Differential Equations*. 1st Edn., Elsevier, Amsterdam, ISBN-10: 0444518320, pp: 523.
- Klimek, K., 2005. Fractional sequential mechanics-models with symmetric fractional derivative. *Czech. J. Phys.*, 55: 1447-1453. DOI: 10.1023/A:1013378221617
- Khalil, H., R.A. Khan, M.H. Al-Smadi and A.A. Freihat, 2015a. Approximation of solution of time fractional order three-dimensional heat conduction problems with Jacobi Polynomials. *J. Math.*, 47: 35-56.
- Khalil, H., R.A. Khan, M.H. Al-Smadi, A.A. Freihat and N. Shawagfeh, 2015b. New operational matrix for shifted Legendre polynomials and fractional differential equations with variable coefficients. *Pun. Uni. J. Math.*, 47: 81-103.

- Khalil, H., R.A. Khan, M. Al-Smadi and A. Freihat, 2015c. A generalized algorithm based on Legendre polynomials for numerical solutions of coupled system of fractional order differential equations. *J. Fract. Calculus Applic.*, 6: 123-143.
- Khader, M.M., N.H. Swetlam and A.M.S. Mahdy, 2014. The chebyshev collection method for solving fractional order Klein-Gordon equation. *Wseas Trans. Math.*, 13: 31-38.
- Komashynska, I. and M. Al-Smadi, 2014. Iterative reproducing kernel method for solving second-order integrodifferential equations of fredholm type. *J. Applied Math.*, 2014: 459509-459519. DOI: 10.1155/2014/45950
- Komashynska, I., M. Al-Smadi, O. Abu Arqub and S. Momani, 2016. An efficient analytical method for solving singular initial value problems of nonlinear systems. *Applied Math. Inform. Sci.*, 10: 647-656. DOI: 10.18576/amis/100224
- Li, Z.B. and J.H. He, 2010. Fractional complex transform for fractional differential equations. *Math. Comput. Applied*, 15: 970-973.
- Mainardi, F., 2010. *Fractional Calculus and Waves in Linear Viscoelasticity*. 1st Edn., World Scientific, ISBN-10: 1908978570, pp: 368.
- Marasi, H.R. and S. Karimi, 2014. Convergence of the variational iteration method for solving fractional Klein-Gordon equation. *J. Math. Comput. Sci.*, 4: 257-266.
- Millar, K.S. and B. Ross, 1993. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. 1st Edn., John Wiley and Sons, Inc., ISBN-10: 0471588849, pp: 384.
- Moaddy, K., S. Momani and I. Hashim, 2011. The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics. *Comput. Math. Applic.*, 61: 1209-1216. DOI: 10.1016/j.camwa.2010.12.072
- Moaddy, K., M. AL-Smadi and I. Hashim, 2015. A novel representation of the exact solution for differential algebraic equations system using residual power-series method. *Discrete Dynam. Nature Society*, 2015: 205207-205218. DOI: 10.1155/2015/205207
- Momani, S., A. Freihat and M. AL-Smadi, 2014. Analytical study of fractional-order multiple chaotic fitzhugh-nagumo neurons model using multistep generalized differential transform method. *Abst. Applied Anal.*, 2014: 276279-276288. DOI: 10.1155/2014/276279
- Odibat, Z. and S. Momani, 2006. Application of variational iteration method to nonlinear differential equations of fractional order. *Int. J. Nonlin. Sci. Numer. Simul.*, 7: 27-34. DOI: 10.1515/IJNSNS.2006.7.1.27
- Ortigueira, M.D., 2010. The fractional quantum derivative and its integral representations. *Commun. Nonlin. Sci. Numer. Simulat.*, 15: 956-962. DOI: 10.1016/j.cnsns.2009.05.026
- Podlubny, I., 1999. *Fractional Differential Equations*. 1st Edn., Academic Press, San Diego, CA, USA.
- Samko, S.G., A.A. Kilbas and O.I. Marichev, 1993. *Fractional Integrals and Derivatives: Theory and Applications*. 1st Edn., Gordon and Breach, New York.
- Secer, A., 2012. Solving time-fractional reaction-diffusion equation by reduced differential transform method. *Int. J. Comp. Sci. Emer. Tech.*, 3: 19-22.
- Sohail, M. and S.T. Mohyud-Din, 2012. Reduced differential transform method for solving a system of fractional PDEs. *Int. J. Mod. Math. Sci.*, 4: 21-29.
- Srivastava, V.K., M.K. Awasthi and M. Tamsir, 2013. RDTM solution of Caputo time fractional-order hyperbolic Telegraph equation. *AIP Adv.*, 3: 032142-032142. DOI: 10.1063/1.4799548
- Srivastava, V.K., M.K. Awasthi and S. Kumar 2014. Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method. *Egypt. J. Basic Applied Sci.*, 1: 60-66. DOI: 10.1016/j.ejbas.2014.01.002
- Sweilam, N.H., M.M. Khader and A.M.S. Mahdy, 2012. On the numerical solution for the linear fractional Klein-Gordon equation using Legendre pseudo-spectral method. *Int. J. Math. Comput. Applic. Res.*, 2: 1-10.
- Yang, A.M., Y.Z. Zhang, C. Cattani, G.N. Xie and M.M. Rashidi *et al.*, 2014. Application of local fractional series expansion method to solve klein-gordon equations on cantor sets. *Abst. Applied Anal.*, 2014: 372741-372746. DOI: 10.1155/2014/372741
- Yusufoglu, E., 2008. The variational iteration method for studying the Klein-Gordon equation. *Applied Math. Lett.*, 21: 669-674. DOI: 10.1016/j.aml.2007.07.023