Journal of Mathematics and Statistics

Research Notes

New Bivariate Wrapped Distributions

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Article history
Received: 12-06-2016
Revised: 10-11-2016
Accepted: 24-11-2016

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Abstract: There are not many models for bivariate directional data. Here, we introduce more than ten new bivariate wrapped distributions. For each distribution, expressions are given for the means, covariances and five correlation coefficients.

Keywords: Characteristic Function, Correlation Coefficients, Directional Data

Introduction

Models for univariate directional data are well established. An excellent review of known models can be found in Mardia and Jupp (2000). However, models for bivariate or multivariate directional data have been limited. A common method for constructing models for directional data is wrapping. We are aware of only the bivariate wrapped normal and bivariate wrapped Cauchy distributions being used as models for bivariate directional data. The bivariate wrapped Cauchy distribution has been used for face analysis (Waine, 2001) among others. The bivariate wrapped normal distribution has been used to model online handwriting recognition (Bahlmann, 2006) among others.

The aim of this paper is to introduce a range of bivariate wrapped distributions and related measures. Let $f(x, y)$ denote a joint pdf of a random vector (X, Y) in $(-\infty,+\infty) \times (-\infty,+\infty)$ with characteristic function $\psi(s, t)$. Then a bivariate wrapped distribution can be defined by the pdf:

$$g(x,y)=\sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} f(x+2\pi j, y+2\pi k)$$

Let m_1 and m_2 denote the marginal means associated with f. Let:

$$z=\begin{bmatrix}\cos(X)\\ \sin(X)\\ \cos(Y)\\ \sin(Y)\end{bmatrix}$$

and let:

$$\mu=\begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{bmatrix}=Ez=\begin{bmatrix}E\cos(X)\\ E\sin(X)\\ E\cos(Y)\\ E\sin(Y)\end{bmatrix}$$

Denote the vector of means and let:

$$S=\begin{bmatrix}s_{1,1} & s_{1,2} & s_{1,2} & s_{1,4}\\ s_{2,1} & s_{2,2} & s_{2,3} & s_{2,4}\\ s_{3,1} & s_{3,2} & s_{3,3} & s_{3,4}\\ s_{4,1} & s_{4,2} & s_{4,3} & s_{4,4}\end{bmatrix}=E[z-\mu][z-\mu]^T$$

$$\begin{bmatrix}
E\{[E\cos(X)-\mu_1]^2\} & E\{[E\cos(X)-\mu_1][E\sin(X)-\mu_2]\}\\
E\{[E\sin(X)-\mu_2][E\cos(X)-\mu_1]\} & E\{[E\sin(X)-\mu_2]^2\}\\
E\{[E\cos(Y)-\mu_3][E\cos(X)-\mu_1]\}\\
E\{[E\cos(Y)-\mu_3][E\sin(X)-\mu_2]\}\\
E\{[E\sin(Y)-\mu_4][E\cos(X)-\mu_1]\}\\
E\{[E\sin(Y)-\mu_4][E\sin(X)-\mu_2]\}\\
E\{[E\cos(X)-\mu_1][E\cos(Y)-\mu_3]\}\\
E\{[E\cos(X)-\mu_1][E\sin(Y)-\mu_4]\}\\
E\{[E\sin(X)-\mu_2][E\cos(Y)-\mu_3]\}\\
E\{[E\sin(X)-\mu_2][E\sin(Y)-\mu_4]\}\\
E\{[E\cos(Y)-\mu_3]^2\} & E\{[E\cos(Y)-\mu_3][E\sin(Y)-\mu_4]\}\\
E\{[E\sin(Y)-\mu_4][E\cos(Y)-\mu_3]\} & E\{[E\sin(Y)-\mu_4]^2\}
\end{bmatrix}$$

Denote the matrix of variances and covariance's. Then:

$$\mu_1 = \mathrm{Re}\left\{\frac{1}{2}\left[\psi(1,0)+\psi(-1,0)\right]\right\},$$
$$\mu_2 = \mathrm{Re}\left\{\frac{1}{2i}\left[\psi(1,0)-\psi(-1,0)\right]\right\},$$
$$\mu_3 = \mathrm{Re}\left\{\frac{1}{2}\left[\psi(0,1)+\psi(0,-1)\right]\right\},$$
$$\mu_4 = \mathrm{Re}\left\{\frac{1}{2i}\left[\psi(0,1)-\psi(0,-1)\right]\right\}$$

and:

$$s_{1,1} = \mathrm{Re}\left\{\frac{1}{4}\left[\psi(2,0)+\psi(-2,0)+2\right]-\mu_1^2\right\},$$
$$s_{1,2} = \mathrm{Re}\left\{\frac{1}{4i}\left[\psi(2,0)-\psi(-2,0)+2\right]-\mu_1\mu_2\right\},$$
$$s_{1,3} = \mathrm{Re}\left\{\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]-\mu_1\mu_3\right\},$$
$$s_{1,4} = \mathrm{Re}\left\{\frac{1}{4i}\left[\psi(1,1)-\psi(1,-1)+\psi(-1,1)-\psi(-1,-1)\right]-\mu_1\mu_4\right\},$$
$$s_{2,2} = \mathrm{Re}\left\{-\frac{1}{4}\left[\psi(2,0)+\psi(-2,0)-2\right]-\mu_2^2\right\},$$
$$s_{2,3} = \mathrm{Re}\left\{\frac{1}{4i}\left[\psi(1,1)+\psi(1,-1)-\psi(-1,1)-\psi(-1,-1)\right]-\mu_2\mu_3\right\},$$
$$s_{2,4} = \mathrm{Re}\left\{-\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]-\mu_2\mu_4\right\},$$
$$s_{3,3} = \mathrm{Re}\left\{\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)+2\right]-\mu_3^2\right\},$$
$$s_{3,4} = \mathrm{Re}\left\{\frac{1}{4i}\left[\psi(0,2)-\psi(0,-2)-2\right]-\mu_3\mu_4\right\},$$
$$s_{4,4} = \mathrm{Re}\left\{-\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)-2\right]-\mu_4^2\right\}$$

where, Re denotes the real part and $i=\sqrt{-1}$ denotes the complex unit. If ψ is a real function then:

$$\mu_1 = \frac{1}{2}\left[\psi(1,0)+\psi(-1,0)\right],$$
$$\mu_2 = 0,$$
$$\mu_3 = \frac{1}{2}\left[\psi(0,1)+\psi(0,-1)\right],$$
$$\mu_4 = 0$$

and:

$$s_{1,1} = \frac{1}{4}\left[\psi(2,0)+\psi(-2,0)+2\right]-\mu_1^2,$$
$$s_{1,2} = 0,$$
$$s_{1,3} = \frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]-\mu_1\mu_3,$$
$$s_{1,4} = 0,$$
$$s_{2,2} = -\frac{1}{4}\left[\psi(2,0)+\psi(-2,0)-2\right]-\mu_2^2,$$
$$s_{2,3} = 0,$$
$$s_{2,4} = -\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]-\mu_2\mu_4,$$
$$s_{3,3} = \frac{1}{4}\left[\psi(0,2)+\psi(0,-2)+2\right]-\mu_3^2,$$
$$s_{3,4} = 0,$$
$$s_{4,4} = -\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)-2\right]-\mu_4^2$$

Let:

$$D = \begin{bmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{bmatrix} = S_{1:2,1:2}^{-1} S_{1:2,3:4} S_{3:4,3:4}^{-1} S_{3:4,1:2}$$

where, $S_{a:b,c:d}$ refers to the submatrix of S with rows a to b and columns c to d.

There are several correlation coefficients between x and y. The ones due to Jammalamadaka and Sarma (1988; Mardia and Puri, 1978; Rivest, 1982; Jupp and Mardia, 1980; Johnson and Wehrly, 1977) are:

$$\rho_{JS} = \frac{SS}{\sqrt{S_x, S_y}} \tag{1}$$

$$\rho_{MP} = \frac{SS^2}{S_x S_y} + \frac{CC^2}{C_x C_y} + \frac{CS^2}{C_x S_y} + \frac{SC^2}{S_x C_y} \tag{2}$$

$$\rho_R = \frac{sign(d_2)\left[T-\sqrt{T^2-4d_2}\right]}{2\max(S_x, S_y)} \tag{3}$$

$$\rho_J = D_{1,1} + D_{2,2} \tag{4}$$

and:

$$\rho_{JW} = \frac{\rho_J + \sqrt{\rho_J^2 - 4d_1}}{2} \tag{5}$$

Respectively, where:

$$d_1 = D_{1,1}D_{2,2} - D_{1,2}D_{2,1}, d_2 = t_{1,1}t_{2,1} - t_{1,2}t_{2,1}, T = t_{1,1} + t_{2,2},$$
$$SS = E\left[\sin(x-m_1)\sin(y-m_2)\right],\ CC = E\left[\cos(x-m_1)\cos(y-m_2)\right],$$
$$CS = E\left[\cos(x-m_1)\sin(y-m_2)\right],\ SC = E\left[\sin(x-m_1)\cos(y-m_2)\right],$$
$$S_x = E\left[\sin^2(x-m_1)\right], S_y = E\left[\sin^2(y-m_2)\right],$$
$$C_x = E\left[\cos^2(x-m_1)\right], C_y = E\left[\cos^2(y-m_2)\right],$$
$$t_{1,1} = E\left[\cos(x)\cos(y)\right],\ t_{1,2} = E\left[\cos(x)\sin(y)\right],$$
$$t_{2,1} = E\left[\sin(x)\cos(y)\right],\ t_{2,2} = E\left[\sin(x)\sin(y)\right]$$

Note that we can write:

$$SS = -\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)+\psi(1,-1)-\psi(-1,1)-\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)-\psi(1,-1)+\psi(-1,1)-\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\}$$
$$+\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\},$$
$$CC = -\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$+\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)-\psi(1,-1)+\psi(-1,1)-\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$+\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)+\psi(1,-1)-\psi(-1,1)-\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\},$$
$$SC = -\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)+\psi(1,-1)-\psi(-1,1)-\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)-\psi(1,-1)+\psi(-1,1)-\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\},$$
$$CS = -\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)-\psi(1,-1)+\psi(-1,1)-\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)+\psi(1,-1)-\psi(-1,1)-\psi(-1,-1)\right]\sin(m_1)\cos(m_2)\right\},$$
$$S_y = -\operatorname{Re}\left\{\frac{1}{4}\left[\psi(2,0)+\psi(0,-2)-2\right]\cos^2(m_2)\right\}$$
$$+\operatorname{Re}\left\{\frac{1}{4}\left[\psi(2,0)+\psi(0,-2)+2\right]\sin^2(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{2i}\left[\psi(2,0)-\psi(0,-2)\right]\sin(m_2)\cos(m_2)\right\},$$
$$C_x = -\operatorname{Re}\left\{\frac{1}{4}\left[\psi(2,0)+\psi(-2,0)+2\right]\cos^2(m_1)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4}\left[\psi(2,0)+\psi(-2,0)-2\right]\sin^2(m_1)\right\}$$
$$+\operatorname{Re}\left\{\frac{1}{2i}\left[\psi(2,0)-\psi(-2,0)\right]\sin(m_1)\cos(m_1)\right\},$$
$$C_y = -\operatorname{Re}\left\{\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)+2\right]\cos^2(m_2)\right\}$$
$$-\operatorname{Re}\left\{\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)-2\right]\sin^2(m_2)\right\}$$
$$+\operatorname{Re}\left\{\frac{1}{2i}\left[\psi(0,2)-\psi(0,-2)\right]\sin(m_2)\cos(m_2)\right\},$$
$$t_{1,1} = \operatorname{Re}\left\{\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\right\},$$
$$t_{1,2} = \operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)-\psi(1,-1)+\psi(-1,1)-\psi(-1,-1)\right]\right\},$$
$$t_{2,1} = \operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)+\psi(1,-1)-\psi(-1,1)-\psi(-1,-1)\right]\right\},$$
$$t_{2,2} = \operatorname{Re}\left\{\frac{1}{4i}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\right\}$$

If ψ is a real function then:

$$SS = -\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\cos(m_2)$$
$$+\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\sin(m_2),$$
$$CC = -\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\cos(m_2)$$
$$-\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\sin(m_2),$$
$$SC = -\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\sin(m_2)$$
$$-\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\cos(m_2),$$
$$CS = -\frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right]\cos(m_1)\sin(m_2)$$
$$-\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]\sin(m_1)\cos(m_2),$$
$$S_x = -\frac{1}{4}\left[\psi(2,0)+\psi(-2,0)-2\right]$$
$$\cos^2(m_1)+\frac{1}{4}\left[\psi(0,2)+\psi(-2,0)+2\right]\sin^2(m_1),$$
$$S_y = -\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)+2\right]$$
$$\cos^2(m_2)+\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)+2\right]\sin^2(m_2),$$
$$C_x = \frac{1}{4}\left[\psi(2,0)+\psi(-2,0)+2\right]$$
$$\cos^2(m_1)-\frac{1}{4}\left[\psi(2,0)+\psi(-2,0)-2\right]\sin^2(m_1)$$
$$C_y = \frac{1}{4}\left[\psi(0,2)+\psi(0,-2)+2\right]$$
$$\cos^2(m_2)-\frac{1}{4}\left[\psi(0,2)+\psi(0,-2)-2\right]\sin^2(m_2)$$
$$t_{1,1} = \frac{1}{4}\left[\psi(1,1)+\psi(1,-1)+\psi(-1,1)+\psi(-1,-1)\right],$$
$$t_{1,2} = 0, t_{2,1} = 0,$$
$$t_{2,2} = -\frac{1}{4}\left[\psi(1,1)-\psi(1,-1)-\psi(-1,1)+\psi(-1,-1)\right]$$

We introduce fourteen bivariate wrapped distributions. They are based on the bivariate stable, bivariate Laplace, bivariate normal, bivariate elliptically symmetric, bivariate Linnik (Anderson, 1992), bivariate Cauchy, bivariate skew normal (Azzalini and Dalla Valle, 1996) and bivariate logistic distributions. For each distribution, we give expressions for μ, S and the five correlation coefficients. These expressions in addition to giving summary measures can also be used for estimating parameters. Codes in the R software

(R Development Core Team, 2016) for computing the given expressions have been developed by the authors.

Because of space restrictions, we present details for only two of the fourteen distributions. The variation of the five correlation coefficients for these two bivariate wrapped distributions is illustrated graphically in section 3. The details for other distributions can be obtained from the corresponding author.

The many distributions introduced could encourage further applications of models for directional data. They could also encourage further models being developed for directional data. A future work is to extend the results in this study to the multivariate case.

The Collection

The two distributions discussed here are the bivariate wrapped Laplace and bivariate wrapped normal distributions.

Bivariate wrapped Laplace distribution with:

$$\psi(s,t)=\left[1+\frac{1}{2}c_{1,1}s^2+\frac{1}{2}c_{2,2}s^2+c_{1,2}st\right]^{-1}$$

For $c_{1,1} > 0$ and $c_{2,2} > 0$, we have:

$$\mu_1=2\left(2+c_{1,1}\right)^{-1},\mu_2=0,\ \mu_3=2\left(2+c_{2,2}\right)^{-1},\mu_4=0,$$

$$s_{1,1}=\frac{c^2_{1,1}\left(c_{1,1}+5\right)}{\left(1+2c_{1,1}\right)\left(2+c_{1,1}\right)^2},s_{1,2}=0,$$

$$s_{1,3}=2\frac{c^2_{1,1}c_{2,2}+c_{1,1}c^2_{2,2}+2c_{2,2}c_{1,1}+8c^2_{1,2}}{\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)\left(2+c_{1,1}\right)\left(2+c_{2,2}\right)},s_{1,4}=0,$$

$$s_{2,2}=\frac{c_{1,1}}{1+2c_{1,1}},s_{2,3}=0,s_{2,4}=\frac{4c_{1,2}}{\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)},$$

$$s_{3,3}=\frac{c^2_{2,2}\left(c_{2,2}+5\right)}{\left(1+2c_{2,2}\right)\left(2+c_{2,2}\right)^2},s_{3,4}=0,s_{4,4}=\frac{c_{2,2}}{1+2c_{2,2}},$$

$$\rho J=4\frac{\left(1+2c_{1,1}\right)\left(c^2_{1,1}+c_{1,1}c^2_{2,2}+2c_{1,1}c_{2,2}+8c^2_{1,2}\right)^2\left(1+2c_{2,2}\right)}{c^2_{1,1}\left(c_{1,1}+5\right)\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)^2\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)^2c^2_{2,2}\left(c_{2,2}+5\right)}$$

$$+16\frac{\left(1+2c_{1,1}\right)c^2_{1,1}\left(1+^2c_{2,2}\right)}{c_{1,1}\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)^2\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)^2c_{2,2}},$$

$$d_1=64\frac{\left(1+2c_{1,1}\right)^2\begin{pmatrix}c^2_{1,1}c_{2,2}+c_{1,1}c^2_{2,2}\\+2c_{1,1}c_{2,2}+8c^2_{1,2}\end{pmatrix}^2\left(1+2c_{2,2}\right)^2c^2_{1,2}}{c3_{1,1}\left(c_{1,1}+5\right)\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)^4\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)4c^3_{2,2}\left(c_{2,2}+5\right)},$$

$$ss=\frac{4c_{1,2}}{\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)},$$

$$s_x=\frac{c_{1,1}}{1+2c_{1,1}},$$

$$s_y=\frac{c_{2,2}}{1+2c_{2,2}},$$

$$\rho_{MP}=\frac{4\left(1+2c_{1,1}\right)\left(1+2c_{2,2}\right)}{\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)^2\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)^2\left(1+c_{1,1}\right)\left(1+c_{2,2}\right)c_{1,1}c_{2,2}}$$

$$.(c^3_{1,1}c_{2,2}+2c^2_{1,1}c^2_{2,2}+4c_{1,1}c^2_{1,2}c_{2,2}+c_{1,1}c^3_{2,2}+4c^2_{1,1}c_{2,2}$$
$$+4c_{1,1}c^2_{1,2}+4c_{1,1}c^2_{2,2}+4c^2_{1,2}c_{2,2}+4c_{1,1}c_{2,2}+4c^2_{1,2})$$

$$\mathrm{T}=2\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)^{-1}$$

$$d_2=8\frac{\left(2+c_{1,1}+c_{2,2}\right)c_{1,2}}{\left(2+c_{1,1}+c_{2,2}-2c_{1,2}\right)^2\left(2+c_{1,1}+c_{2,2}+2c_{1,2}\right)^2}$$

Bivariate wrapped normal distribution with:

$$\psi(s,t)=e^{-\frac{1}{2}c_{1,1}s^2-\frac{1}{2}c_{2,2}t^2-c_{1,2}st}$$

For $c_{1,1} > 0$ and $c_{2,2} > 0$, we have:

$$\mu_1=e^{-\frac{1}{2}c1,1},\mu_2=0,\mu_3=e^{--\frac{1}{2}c2,2},\mu_4=0,$$

$$s_{1,1}=\frac{1}{2}e^{-2c1,1}+\frac{1}{2}-e^{-c1,1},\ s_{1,2}=0,$$

$$s_{1,3}=\frac{1}{2}e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2}+\frac{1}{2}e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}-e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2},$$

$$s_{1,4}=0,s_{2,2}=-\frac{1}{2}e^{-2c1,1}+\frac{1}{2},s_{2,3}=0,$$

$$s_{2,4}=\frac{1}{2}e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}-\frac{1}{2}e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2},$$

$$s_{3,3}=\frac{1}{2}e^{-2c2,2}+\frac{1}{2}-e^{-c2,2},s_{3,4}=0,s_{4,4}=-\frac{1}{2}e^{-2c2,2}+\frac{1}{2},$$

$$\rho_J=\frac{\left[-2e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2}+e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2}+e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}\right]^2}{\left[-2e^{-c1,1}+e^{-2c1,1}+1\right]\left[-2e^{-c2,2}+e^{-2c2,2}+1\right]}$$
$$+\frac{\left[e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2}-e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}\right]^2}{\left[e^{-2c1,1}-1\right]\left[e^{-2c2,2}-1\right]},$$

$$d=\frac{\left[2e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2}-e^{-\frac{1}{2}c1,1-c2,2=c1,2}-e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}\right]^2}{\left[2e^{-c1,1}-e^{-2c1,1}-1\right]\left[2e^{-c2,2}-e^{-2c2,2}-1\right]}$$
$$.\frac{\left[e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2}-e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}\right]^2}{\left[e^{-2c1,1}-1\right]\left[\left[e^{-2c2,2}-1\right]\right]},$$

$$SS=\frac{1}{2}e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}-\frac{1}{2}e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2},$$

$$S_x = \frac{1}{4}\left[2 - 2e^{-2ec1,1}\right],$$

$$S_y = \frac{1}{4}\left[2 - 2e^{-2c2,2}\right],$$

$$\rho_{MP} = \frac{\left[e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2} + e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}\right]^2}{\left[e^{-2c1,1}+1\right]\left[e^{-2c2,2}+1\right]}$$

$$+\frac{\left[e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2-c1,2} - e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2}\right]^2}{\left[e^{-2c1,1}-1\right]\left[e^{-2c2,2}-1\right]},$$

$$\mathrm{T} = e^{-\frac{1}{2}c1,1-\frac{1}{2}c2,2+c1,2},$$

$$d_2 = \frac{1}{4}e^{-c1,1-c2,2+2c1,2} - \frac{1}{4}e^{-c1,1-c2,2-2c1,2}.$$

Discussion

Here, we produce graphs that show how the correlation coefficients in (1) to (5) vary with respect to parameters of the two bivariate wrapped distributions in section 2.

Figure 1 shows how (1) to (5) vary versus ρ for the standard bivariate wrapped Laplace distribution. Figure 2 shows how (1) to (5) vary versus ρ for the standard bivariate wrapped normal distribution.

Both figures show similar pattern. Jammalamadaka and Sarma (1988) correlation coefficient is an increasing function of ρ taking the values −1 and 1 when $\rho = -1$ and $\rho = 1$, respectively. Rivest (1982) correlation coefficient is either an increasing or a decreasing function of ρ taking the values 1 and −1 at the extremes of ρ. Mardia and Puri (1978; Jupp and Mardia, 1980; Johnson and Wehrly, 1977) correlation coefficients all exhibit a parabolic shape symmetric around $\rho = 0$. Mardia and Puri (1978; Jupp and Mardia, 1980) correlation coefficients take the maximum value of 2. Johnson and Wehrly (1977) correlation coefficient takes the maximum value of 1. The minimum value taken by (Mardia and Puri, 1978; Jupp and Mardia, 1980; Johnson and Wehrly, 1977) correlation coefficients is zero.

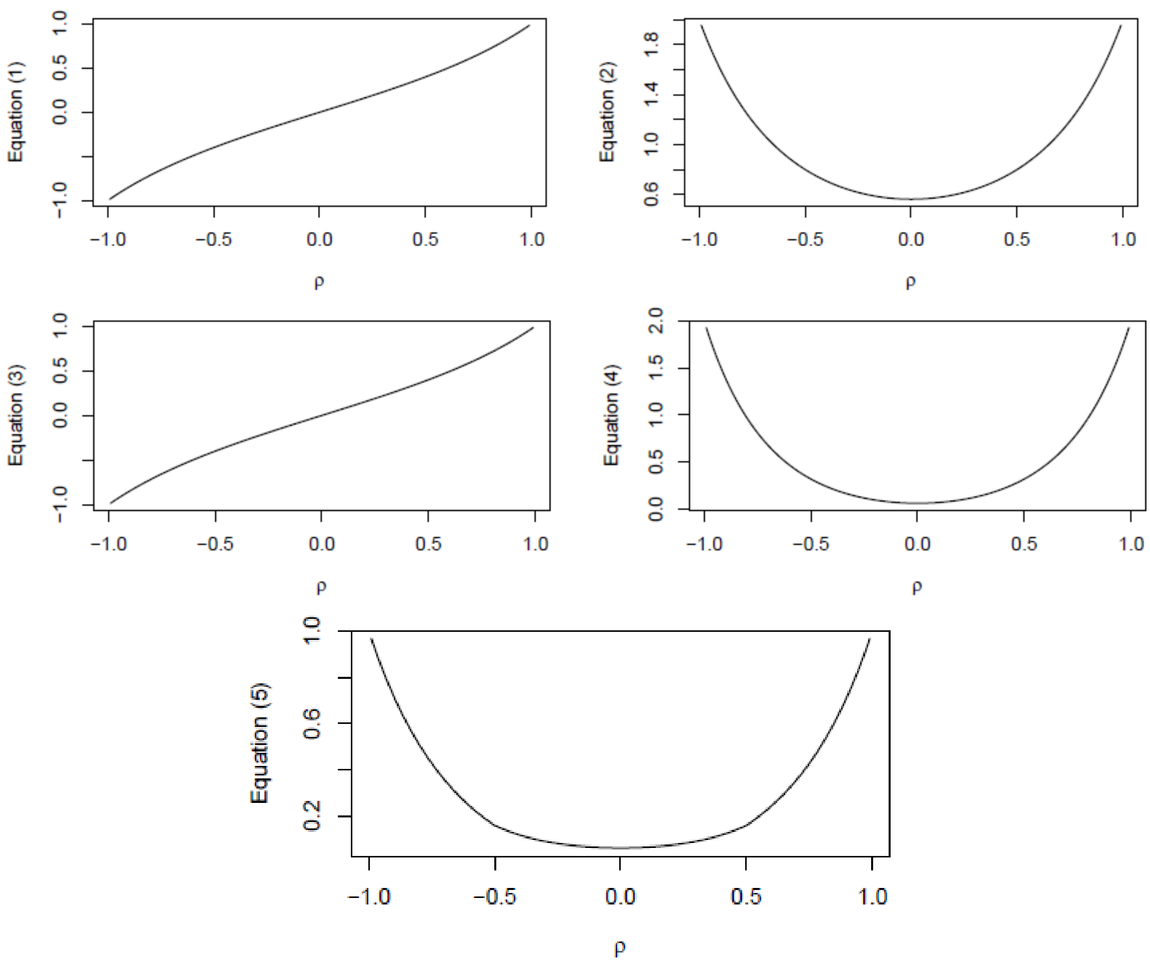

Fig. 1. Correlation coefficients (1) to (5) versus ρ for the standard bivariate wrapped Laplace distribution

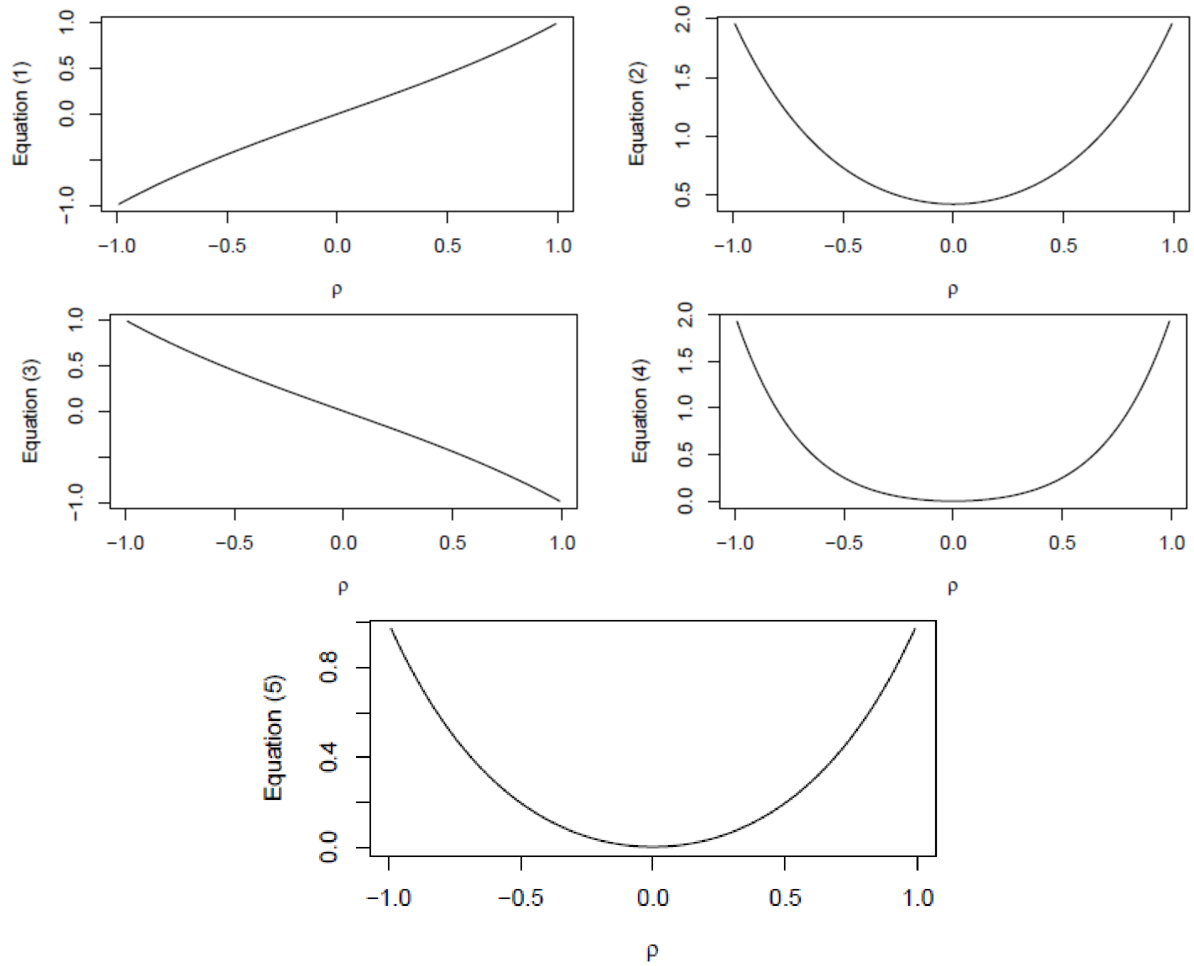

Fig. 2. Correlation coefficients (1) to (5) versus ρ for the standard bivariate wrapped normal distribution

For illustration, we have considered only the two distributions discussed in section 2. But the variation of the five correlation coefficients with respect to the parameters of the other distributions was similar to those variations reported in Fig. 1 and 2.

Acknowledgment

The authors would like to thank the Editor and the referee for careful reading and comments which greatly improved the paper.

Author’s Contributions

S. Nadarajah: Wrote section 1-2.
Y. Zhang: Wrote section 2.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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