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Characterizations for the Power Inverse Gaussian and Generalized Inverse Gaussian Distributions Based on Conditional Moments

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Abstract: In this study, we characterize the power inverse Gaussian and generalized inverse Gaussian distributions based on recurrence relations for conditional moments in terms of hazard rate functions. Relations for conditional moments of the largest order statistic (upper record value) given the second largest order statistic (upper record value) are derived from these recurrence relations. Characterizations of the inverse Gaussian, reciprocal inverse Gaussian, gamma and reciprocal gamma distributions are obtained as special cases of both the power inverse Gaussian and the generalized inverse Gaussian distributions.

Keywords: Characterization, Power Inverse Gaussian Distribution, Generalized Inverse Gaussian Distribution, Conditional Moments, Mean Residual Life, Order Statistics, Record Values

Introduction

The random variable (rv) X is said to have the Power Inverse Gaussian (PIG) distribution with probability density function (pdf) f(x) and distribution function (df) F(x) if its pdf is given for x>0, by:

$$f(x) \equiv f(x;v,\mu,c) = \frac{1}{c\mu\sqrt{2\pi}}(x/\mu)^{-(1+v/2)} exp\left\{-\frac{1}{2(vc)^2}[(x/\mu)^{v/2}-(x/\mu)^{-v/2}]^2\right\} \quad (1.1)$$

where, μ, c>0 and $\nu \neq 0$. When $\nu = 1$ and $\nu = -1$, the PIG reduces respectively to the Inverse Gaussian (IG) distribution and the Reciprocal Inverse Gaussian (RIG) distribution.

We can show that the pdf (1.1) satisfies the following differential equation:

$$2vc^2xf'(x)+[(x/\mu)^v-(x/\mu)^{-v}+vc^2(v+2)]f(x)=0 \quad (1.2)$$

A rv X is said to have the Generalized Inverse Gaussian (GIG) distribution with pdf g(x) and df G(x) if its pdf is given for x>0, by:

$$g(x) \equiv g(x;a,b,p) = Ax^{p-1}exp\left(-ax-\frac{b}{x}\right) \quad (1.3)$$

where, A is the normalizing constant.

We will consider the following three cases:

- a>0; b>0 and p any real number, this case gives the IG when p = -1/2
- a>0; b = 0 and p>0, we obtain the Gamma Distribution (GD)
- a = 0; b>0 and p<0, we obtain the reciprocal gamma distribution

A rv X is distributed as RIG (or reciprocal gamma) distributed, if and only if 1/X is IG (or gamma) distributed.

The pdf (1.3) satisfies the following differential equation:

$$g'(x) = g[(a-b/x^2)-(p-1)/x]g(x) = 0 \quad (1.4)$$

Characterization problems arise naturally in areas such as reliability, statistical inference and model building where one is interested in knowing whether a particular hypothesis or model is equivalent to some other hypothesis or model that is appealing in some sense. Characterization theorems declare that the existence of a property determines the distribution function, possibly within some family [for more details in characterization theory see, Galambos and Kotz (1978) and Arnold *et al.* (2008)]. Among others, Shanbhag (1970), Telcs *et al.* (1985), Osaki and Li (1988), Nair and Sankaran (1991), Consul (1995), Ahmad (1996), Seshadri and Wesolowski (2001) and Chou and Huang (2004) have characterized some distributions based on conditional moments.

Iwase and Hirano (1990) discussed the PIG and its applications. The PIG has characterized by Kawamura and Iwase (2003). Among others, Latac and Seshadri (1983), Seshadri and Wesolowski (2001) and Chou and Huang (2004) have characterized the GIG [for more details in GIG, see Seshadri (1993)]. Many studies about inverse Gaussian and generalized Gaussian distributions have been carried out in different hypotheses, for examples, Balakrishnan *et al*. (2009) studied the mixture inverse Gaussian distributions and its transformations, moments and applications statistics.

Zhu and Joe (2009) proposed the modelling heavy-tailed count data using a generalized Poisson-inverse Gaussian family. Rigby *et al*. (2008) illustrated a framework for modelling over dispersed count data, including the Poisson-shifted generalized inverse Gaussian distribution. Marchetti and Mudholkar (2002) presented the characterization theorems and goodness of-fit tests. Also, Mudholkar and Tian (2002) investigated an entropy characterization of the inverse Gaussian distribution and related goodness -of-fit test. Woldie *et al*. (2001) displayed the function for Inverse Gaussian Regression Models. Recently, inverse Gaussian distribution or process is used for many applications, see Chen *et al*. (2015), Liu *et al*. (2014) and Peng *et al*. (2014).

In this study, we are concerned with characterizing the PIG and GIG based on recurrence relations for conditional moments. The characterizing conditions are in terms of the hazard rate function (hrf). Characterizations for IG, RIG and GD have also derived as they are special cases of PIG and GIG.

Characterizations of PIG and GIG Distributions

In this section, we make use of the two characterizing differential equations (1.2) and (1.4) to derive the following two characterizations for PIG and GIG.

Theorem 2.1. Let X be a rv with a pdf equal to f(x) and df equal to F(x), then, for j = 0,... and y $\geq$ 0, the recurrence relation:

$$\begin{aligned} &E(X^{j+v}\mid X\geq y)=vc^2\mu^v(2j-v)E(X^j\mid X\geq y)\\ &+\mu^{2v}E(X^{j-v}\mid X\geq y)+2vc^2\mu^v y^{j+1}H(y)\end{aligned} \quad (2.1)$$

is satisfied, if and only if X has the pdf (1.1), where $H(.)=f(.)/\bar{F}(.)$ is the hrf, corresponding to the pdf (1.1), and $\bar{F}(.)=1-F(.)$

Proof. It is well known that:

$$E(X^j\mid X\geq y)=\frac{1}{\bar{F}(y)}\int_y^\infty x^j f(x)dx \quad (2.2)$$

So, if X has the pdf (1:1), then we have:

$$\begin{aligned} &E(X^j\mid X\geq y)=\frac{\mu^{v/2}}{c\sqrt{2\pi}\bar{F}(y)}\int_y^\infty x^{j-v/2-1}\\ &exp\left\{-\frac{1}{2(vc)^2}\left[(x/\mu)^{v/2}-(x/\mu)^{-v/2}\right]^2-\right]\right\}\Big]dx\\ &=\frac{\mu^{v/2}}{c\sqrt{2\pi}\bar{F}(y)}\int_y^\infty exp\left\{\begin{matrix}-\frac{1}{2(vc)^2}[(x/\mu)^{v/2}\\ -(x/\mu)^{-v/2}]^2\end{matrix}\right\}\times d\left[-\frac{x^{j-v/2}}{(v/2)-j}\right]\end{aligned}$$

Integrating by parts, we have:

$$\begin{aligned} &E(X^j\mid X\geq y)=\frac{2_y^{j+1}}{2j-v}H(y)+\frac{1}{vc^2\mu^v(2j-v)\bar{F}(y)}\\ &\int_y^\infty x^{j+v}\frac{1}{c\mu\sqrt{2\pi}}(x/\mu)^{-v/2-1}\times exp\left\{\begin{matrix}-\frac{1}{2(vc)^2}[(x/\mu)^{v/2}\\ -(x/\mu)^{-v/2}]^2\end{matrix}\right\}dx\\ &-\frac{\mu^v}{vc^2(2j-v)\bar{F}(y)}\int_y^\infty x^{j-v}\frac{1}{c\mu\sqrt{2\pi}}(x/\mu)^{-v/2-1}\\ &\times exp\left\{-\frac{1}{2(vc)^2}[(x/\mu)^{v/2}-(x/\mu)^{-v/2}]^2\right\}dx\end{aligned}$$

After making use of (2.2) and rearranging the terms we have the desired result (2.1).

Conversely, if the characterizing condition (2.1) is satisfied then:

$$\begin{aligned} &\int_y^\infty x^{j+v}f(x)dx=vc^2\mu^v(2j-v)\int_y^\infty x^j f(x)dx\\ &+\mu^{2v}\int_y^\infty x^{j-v}f(x)dx+2vc^2\mu^v y^{j+1}f(y)\end{aligned} \quad (2.3)$$

Differentiating both sides of (2.3) with respect to y, we obtain the differential equation:

$$2vc^2y\,f'(y)+[(y/\mu)^v-(y/\mu)^{-v}+vc^2(v+2)]f(y)=0$$

Whose solution is the pdf (1.1).

Next, we can prove the following characterization for the pdf (1.3).

Theorem 2.2. For j = 0,1,... and y $\geq$ 0, the following recurrence relation is satisfied:

$$\begin{aligned} &E(X^{j+1}\mid X\geq y)=\frac{j+p}{a}E(X^j\mid X\geq y)\\ &+\frac{b}{a}E(X^{j-1}\mid X\geq y)+\frac{j^{y+1}}{a}h(y),\end{aligned} \quad (2.4)$$

if and only if the rv X has the pdf (1.2), where $h(.)=g(.)/\bar{G}(.)$ is the hrf, corresponding to the pdf (1.3), and $\bar{G}(.)=1-G(.)$.

Proof. Using (1.4) and (2.2), the proof is similar to that of Theorem 1.

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Remarks

Let X1:n ≤...≤$X_{n:n}$ and $X_{(1)}$ ≤...≤ $X_{(m)}$, be the order statistics and the upper record values based on a random sample of size n given as X_1,...,X_n, respectively.

It is well known that:

$$E(X^j|X \geq y) = E(X^j_{n;n}|X_{n-1:n} = y) = E(X^j_m|X_{m-1} = y)$$

So, from (2.1), we can obtain the following recurrence relations for conditional moments of PIGD:

$$E(X^{j+v}_{n;n}|X_{n-1:n} = y) = vc^2\mu^v(2j-v)E(X^j_{n:n}|X_{n-1:n} = y) + \mu^{2v}E(X^{j-v}_{n:n}|X_{n-1:n} = y) + 2vc^2\mu^v y^{j+1}H(y) \quad (2.5)$$

And:

$$E(X^{j+v}_m|X_{(m-1)} = y) = vc^2\mu^v(2j-v)E(X^j_m|X_{(m-1)} = y) + \mu^{2v}E(X^{j-v}_{(m)}|X_{(m-1)} = y) + 2vc^2\mu^v y^{j+1}H(y) \quad (2.6)$$

From (2.4), we obtain the following recurrence relations for conditional moments of GIG:

$$E(X^{j+1}_{n:n}|X_{n-1:n} = y) = \frac{j+p}{a}E(X^j_{n:n}|X_{n-1:n} = y) + \frac{b}{a}E(X^{j-1}_{n:n})|X_{n-1:n} = y) + \frac{y^{j+1}}{a}h(y) \quad (2.7)$$

And:

$$E(X^{j+1}_{(m)}|X_{(m-1)} = y) = \frac{j+p}{a}E(X^j_{(m)}|X_{(m-1)} = y) + \frac{b}{a}E(X^{j-1}_{(m)})|X_{(m-1)} = y) + \frac{y^{j+1}}{a}h(y) \quad (2.8)$$

(2) If we put y = 0 in (2.1) and (2.4), we obtain the following recurrence relations for moments of PIG and GIG, respectively:

$$E(X^{j+v}) = vc^2\mu^v(2j-v)E(X^j) + \mu^{2v}E(X^{j-v}) \quad (2.9)$$

And:

$$E(X^{j+1}) = \frac{j+p}{a}E(X^j) + \frac{b}{a}E(X^{j-1}) \quad (2.10)$$

Special Cases

In this section, we will specialize the characterizing conditions (2.1) and (2.4) for PIG and GIG to the IG, RIG, gamma and reciprocal gamma distributions.

Inverse Gaussian Distribution

- Putting v = 1 and $\mu = \lambda c^2$ ($\lambda > 0$), in (1.1), we obtain the IG in the form:

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left[-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right] \quad (3.1)$$
$$x > 0, (\mu > 0, \lambda > 0)$$

And the characterizing condition (2.1) reduces to:

$$E(X^{j+1}|X \geq y) = \frac{\mu^2(2j-1)}{\lambda}E(X^j|X \geq y) + \mu^2 E(X^{j-1}|X \geq y) + \frac{2\mu^2 y^{j+1}}{\lambda}H(y) \quad (3.2)$$

From (2.5) and (2.6), we obtain:

$$E(X^{j+1}_{n:n}|X_{n-1:n} = y) = \frac{\mu^2(2j-1)}{\lambda}E(X^j_{n:n}|X_{n-1:n} = y) + \mu^2 E(X^{j-1}_{n:n}|X_{n-1:n} = y) + \frac{2\mu^2 y^{j+1}}{\lambda}H(y)$$

And:

$$E(X^{j+1}_{(m)}|X_{(m-1)} = y) = \frac{\mu^2(2j-1)}{\lambda}E(X^j_{(m)}|X_{(m-1)} = y) + \mu^2 E(X^{j-1}_{(m)}|X_{m-1)} = y) + \frac{2\mu^2 y^{j+1}}{\lambda}H(y)$$

In this case, the higher moments of IG when j = 2,3,... can be obtained, from (2.9), by:

$$E(X^{j+1}) = \frac{\mu^2(2j-1)}{\lambda}E(X^j) + \mu^2 E(X^{j-1})$$

Since the first and the second moments of IG are μ and $\mu^3/\lambda + \mu^2$, respectively.

- Putting p = -1/2, a = $\lambda/2\mu^2$ and b = $\lambda/2$, in (1.3) (hence A = $(\lambda/2\pi)^{1/2}$ exp(λ/μ))

We obtain the IG in the form (3.1) and the characterizing condition (2.4) takes the form (3.2) with h(y) = H(y).

If we put μ = 1 in (3.1), we obtain Wald distribution which can be characterized by (3.2) after putting μ = 1.

Reciprocal Inverse Gaussian Distribution

- Putting v = -1 and $\mu = 1/\lambda c^2$ in (1.1), we obtain the RIG in the form:

$$f(x) = \left(\frac{\lambda}{2\pi x}\right)^{1/2} exp\left[-\frac{\lambda x}{2\mu^2}\left(\frac{1}{2}-\mu\right)^2\right] \quad (3.3)$$
$$x > 0, (\mu, >, 0, \lambda > 0)$$

And the characterizing condition (2.1) reduces to:

$$E(X^{j+1}|X \geq y) = \frac{2j+1}{\lambda} E(X^j|X \geq y) + \mu^2 E(X^{j-1}|X \geq y) + \frac{2y^{j+1}}{\lambda} H(y) \quad (3.4)$$

The recurrence relations (2.5), (2.6) and (2.9) reduce to:

$$E(^{j+1}_{n:n}|X_{n-1:n} = y) = \frac{2j+1}{\lambda} E(X^j_{n:n}|X_{n-1:n} = y) + \mu^2 E(X^{j-1}_{n:n}|X_{n-1;n} = y) + \frac{2y^{j+1}}{\lambda} H(y)$$

$$E(^{j+1}_{(m)}|X_{(m-1)} = y) = \frac{2j+1}{\lambda} E(X^j_{(m)}|X_{(m-1)} = y) + \mu^2 E(X^{j-1}_{n:n}|X_{n-1;n} = y) + \frac{2y^{j+1}}{\lambda} H(y)$$

And:

$$E(X^{j+1}) = \frac{2j+1}{\lambda} E(X^j \mu^2 E(X^{j-1})$$

- Putting p = 1/2, a = λ = 2 and b = $\lambda/2\mu^2$, in (1.3) (hence A = $(\lambda/2\pi)^{1/2}$ exp(λ = 2μ))

We obtain the RIG in the form (3.3) and the characterizing condition (2.4) reduces to (3.4) with h(y) = H(y).

Gamma Distribution

Letting p>0, a>0 and b = 0, in (1.3) (hence A = $a^p/\Gamma(p)$), we obtain the gamma distribution in the form:

$$g(x) = \frac{a^p}{\Gamma(p),} x^{p-1} exp(-ax) x > 0 \quad (3.5)$$

The recurrence relations (2.4), (2.7), (2.8) and (2.10) reduce to:

$$E(X^{j+1}|X \geq y) = \frac{j+p}{a} E(X^j|X \geq y) + \frac{y^{j+1}}{a} h(y),$$
$$E(X^{j+1}_{n:n}|X_{n-1:n=y}) = \frac{j+p}{a} E(X^j_{(m)}|X_{n-1:n} = y) + \frac{y^{j+1}}{a} h(y), \quad (36)$$
$$E(X^{j+1}_m|X_{(m-1)} = y) = \frac{j+p}{a} E(X^j_{(m)}|X_{(m-1)} = y) + \frac{y^{j+1}}{a} h(y)$$

And:

$$E(X^{j+1}) = \frac{j+p}{a} E(X^j)$$

Or, equivalently, for j = 1; 2,...:

$$E(X^j) = \frac{\Gamma(p+j)}{a^j \Gamma(p)} \quad (3.7)$$

If we choose p = 1 in (3.5), we obtain the exponential distribution. Equation (2.10), after using h(y) = a, reduces then to:

$$E(X^{j+1}|X \geq y) = \frac{j+1}{a} E(X^j|X \geq y) + y^{j+1}$$

Which for j = 0, reduces to a result of Shanbhag (1970).

If we choose j = 0, in (3.6), we get the following result of Osaki and Li (1988) for GD:

$$E(X|X \geq y) = \frac{p}{a} + \frac{y}{a} h(y)$$

Remarks

It is well known that the residual life for a unit at age y ≥ 0 is T ≡ X-y and the mean residual life (MRL) for a unit at age y≥0 is:

$$E(T|T \geq y) = E(X|X \geq) - y$$

The characterizations derived above can be applied to practical problems. For example, in the case when j = 0 in the characterizing condition (3.7), if we want to check the randomness of a given sample of positive real numbers x truncated on the left at a given y, we begin by calculating the expected value of X based on the truncated sample and we then compare it to $\frac{p}{a} + \frac{y}{a} h(y)$ at the estimates of p; a and h(y).

Conclusion

The main concern of this work is that to characterize the power inverse Gaussian, and the generalized inverse Gaussian distributions based on recurrence relations for conditional moments in terms of hazard rate functions. Characterization problems arise naturally in many different areas like reliability, statistical inference and model building. The two characterizations for PIG and GIG have been derived from the power inverse Gaussian distribution. As special cases of both the power inverse Gaussian and the generalized inverse Gaussian distributions, the Characterizations of the inverse Gaussian, reciprocal inverse Gaussian, gamma and reciprocal gamma distributions are obtained

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Ethics

This article is original and contains unpublished material. In the same time I confirm that no ethical issues are involved.

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