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LOCAL FRACTIONAL VARIATIONAL ITERATION METHOD FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITHIN LOCAL FRACTIONAL OPERATORS

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ABSTRACT

The paper uses the Local fractional variational Iteration Method for solving the second kind Volterra integro-differential equations within the local fractional integral operators. The analytical solutions within the non-differential terms are discussed. Some illustrative examples will be discussed. The obtained results show the simplicity and efficiency of the present technique with application to the problems for the integral equations.

Keywords: Local Fractional Variational Iteration Method, Local Fractional Operator, Local Fractional Volterra Integro-Differential Equation

1. INTRODUCTION

The theory of local fractional calculus is one of useful tools to process the fractal and continuously non differentiable functions (Kolwankar and Gangal, 1998; He, 2011; He *et al*., 2012; Parvate and Gangal, 2009; Carpinteri *et al*., 2004; Yang, 2011a; 2011b; 2011c). It was successfully applied in local fractional Fokker-Planck equation (Kolwankar and Gangal, 1998), the fractal heat conduction equation (He, 2011; Yang, 2011c), fractal-time dynamical systems (Parvate and Gangal, 2009), fractal elasticity (Carpinteri *et al*., 2004), local fractional diffusion equation (Yang, 2011c), local fractional Laplace equation (Yang, 2011b; 2012a), local fractional integral equations (Yang, 2012b; 2012c; 2012d), local fractional differential equations (Yang, 2012e; Zhong *et al*., 2012; Zhong and Gao, 2011), fractal wave equation (Yang, 2011b; 2012a; Yang and Baleanu, 2012).

Recently, the local fractional variational iteration method (Yang and Baleanu, 2012) is derived from local fractional operators (Yang, 2011a; 2011b; 2011c; 2012a; 2012b; 2012c; 2012d; 2012e; Zhong *et al*., 2012; Zhong and Gao, 2011). The method, which accurately computes the solutions in a local fractional series form or in an exact form, presents interest to applied sciences for problems where the other methods cannot be applied properly.

This study is organized as follows. In section 2, the basic mathematical tools are reviewed. Section 3 presents the local fractional variational iteration method based on local fractional operator. Illustrative examples is shown in section 4. Conclusions are in section 5.

2. PRELIMINARY DEFINITIONS

In this section, we recall briefly some basic theory of local fractional calculus and for more details, (Yang and Baleanu, 2012; Su *et al*., 2013; Yang *et al*., 2013a; 2013b; 2013c; Yang, 2012f; Wang *et al*., 2014; Yang *et al*., 2013d; Kilbas *et al*., 2006; Ma *et al*., 2013; Yang *et al*., 2013e; 2013f; 2013g).

Definition 1

Suppose that there is the relation Equation 2.1:

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \ , 0 < \alpha \le 1 \tag{2.1}$$

With $|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$, then the function $f(x)$ is called local fractional continuous at $x=x_0$ and it is denoted by $\lim_{x\to x_0} f(x)=f(x_0)$.

Definition 2

Suppose that the function $f(x)$ satisfies condition (2.1), for $x\in(a,b)$; it is so called local fractional continuous on the interval (a,b), denoted by $f(x)\in C_\alpha(a,b)$.

Definition 3

In fractal space, let $f(x)\in C_\alpha(a,b)$, local fractional derivative of $f(x)$ of order α at $x=x_0$ is given by Equation 2.2:

$$\begin{aligned} D_x^\alpha f(x_0) &= \frac{d^\alpha}{dx^\alpha} f(x)\Big|_{x=x_0} \\ &= f^{(\alpha)}(x)=\lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha} \end{aligned} \quad (2.2)$$

where, $\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(\alpha+1)\Delta(f(x)-f(x_0))$.

Local fractional derivative of high order is written in the form Equation 2.3:

$$f^{(k\alpha)}(x)=\frac{d^{k\alpha}}{dx^{k\alpha}}f(x)=\overbrace{D_x^\alpha D_x^\alpha \ldots\ldots D_x^\alpha}^{k\,times} f(x). \quad (2.3)$$

Definition 4

A partition of the interval $[a,b]$ is denoted as $(t_j,t_{j+1}),\ j=0,\ldots,N-1,\ t_0=a$ and $t_N=b$ with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_0,\Delta t_1,\ldots..\}$. Local fractional integral of $f(x)$ in the interval $[a,b]$ is given by Equation 2.4:

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha \end{aligned} \quad (2.4)$$

Note: If the functions are local fractional continuous then the local fractional derivatives and integrals exist. Some properties of local fractional derivative and integrals are given in (Yang, 2012f).

Definition 5

In fractal space, the Mittage Leffler function, sine function and cosine function are, respectively Equation 2.5 to 2.7:

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)},\ 0<\alpha\le 1 \quad (2.5)$$

$$\sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]},\ 0<\alpha\le 1 \quad (2.6)$$

$$\cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k\alpha}}{\Gamma[1+2k\alpha]},\ 0<\alpha\le 1 \quad (2.7)$$

3. ANALYSIS OF THE METHOD

The standard *ka* order local fractional Volterra integro-differential equation of the second kind is given by:

$$u^{(k\alpha)}(x)=f(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x K(x,t)u(t)(dt)^\alpha \quad (3.1)$$

where, $u^{(k\alpha)}(x)=\frac{d^{k\alpha}u(x)}{dx^{k\alpha}}$ and $u(0)=a_0,\ u^{(\alpha)}(0)=a_1,\ u^{(2\alpha)}(0)=a_2,\ldots,u^{((k-1)\alpha)}(0)=a_{k-1}$ are the initial conditions.

According to the rule of local fractional variational iteration method, the correction local fractional functional for Equation 3.1 is given by Equation 3.2:

$$u_{n+1}(x)=u_n(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)}\left[u_n^{(k\alpha)}(\zeta)-f(\zeta)-\frac{1}{\Gamma(1+\alpha)}\int_0^\zeta K(\zeta,r)u_n(r)(dr)^\alpha\right](d\zeta)^\alpha \quad (3.2)$$

where, $\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)}$ is a general fractal Lagrange's multiplier.

Here, we can construct a correction functional as follows Equation 3.3:

$$u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)}\left[u_n^{(k\alpha)}(\zeta) - f(\zeta) - \frac{1}{\Gamma(1+\alpha)}\int_0^\zeta K(\zeta,r)\tilde{u}_n(r)(dr)^\alpha\right](d\zeta)^\alpha \tag{3.3}$$

where, $\tilde{u}_n$ is considered as a restricted local fractional variation; that is, $\delta^\alpha \tilde{u}_n = 0$, we obtain the following fractal Lagrange multiplier Equation 3.4:

$$\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} = (-1)^k \frac{(\zeta - x)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} \tag{3.4}$$

Therefore Equation 3.5 and 3.6:

$$u_0(x) = u(0) + \frac{x^\alpha}{\Gamma(1+\alpha)}u^{(\alpha)}(0) + \cdots + \frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]}u^{((k-1)\alpha)} \tag{3.5}$$

$$u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)}\int_0^x (-1)^k \frac{(\zeta - x)^{(k-1)\alpha}}{\Gamma(1+(k-1))}\left[u_n^{(k\alpha)}(\zeta) - f(\zeta) - \frac{1}{\Gamma(1+\alpha)}\int_0^\zeta K(\zeta,r)u_n(r)(dr)^\alpha\right](d\zeta)^\alpha, n \ge 0. \tag{3.6}$$

Finally, the solution is Equation 3.7:

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{3.7}$$

4. ILLUSTRATIVE EXAMPLES

In this section three examples for the local fractional Volterra integro-differential equation is presented in order to demonstrate the simplicity and the efficiency of the above method.

Example 1

We consider the local fractional Volterra integro-differential Equation 4.1:

$$u^{(\alpha)}(x) = 1 + \frac{1}{\Gamma(1+\alpha)}\int_0^x u(t)(dt)^\alpha \quad ,u(0) = 1 \tag{4.1}$$

The correction functional for this Equation 4.2 is given by:

$$u_{n+1}(x) = u_n(x) - \frac{1}{\Gamma(1+\alpha)}\int_0^x \left[u_n^{(\alpha)}(\zeta) - 1 - \frac{1}{\Gamma(1+\alpha)}\int_0^\zeta u_n(r)(dr)^\alpha\right](d\zeta)^\alpha \tag{4.2}$$

where, we used $\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} = -1$ for first-order integro-differential equation as shown in (3.4).

We can use the initial condition to select $u_0(x) = u(0) = 1$. Using this selection into the correction functional gives the following successive approximations Equation 4.3 to 4.7:

$$u_0(x) = 1 \tag{4.3}$$

$$u_1(x) = u_0(x) - \frac{1}{\Gamma(1+\alpha)}\int_0^x \left[u_0^{(\alpha)}(\zeta) - 1 - \frac{1}{\Gamma(1+\alpha)}\int_0^\zeta u_0(r)(dr)^\alpha\right](d\zeta)^\alpha = 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \tag{4.4}$$

$$u_2(x) = u_1(x) - \frac{1}{\Gamma(1+\alpha)}\int_0^x \left[u_1^{(\alpha)}(\zeta) - 1 - \frac{1}{\Gamma(1+\alpha)}\int_0^\zeta u_1(r)(dr)^\alpha\right](d\zeta)^\alpha = 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \tag{4.5}$$

$$u_3(x) = u_2(x) - \frac{1}{\Gamma(1+\alpha)}\int_0^x \left[u_2^{(\alpha)}(\zeta) - 1 - \frac{1}{\Gamma(1+\alpha)}\int_0^\zeta u_2(r)(dr)^\alpha\right](d\zeta)^\alpha = 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{x^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{x^{6\alpha}}{\Gamma(1+6\alpha)} \tag{4.6}$$

And so on:

$$u_n(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots+\frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)}$$
$$u(x)=\lim_{n\to\infty}u_n(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\cdots \quad (4.7)$$

That gives the exact solution Equation 4.8:

$$u(x)=E_{\alpha}(x^{\alpha}). \quad (4.8)$$

Example 2

We consider the local fractional Volterra integro-differential Equation 4.9:

$$u^{(2\alpha)}(x)=1+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha}\quad ,u(0)=1,\ u^{(\alpha)}(0)=0. \quad (4.9)$$

The correction functional for this Equation 4.10 is given by:

$$u_{n+1}(x)=u_n(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_n^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_n(r)(dr)^{\alpha}\right](d\zeta)^{\alpha} \quad (4.10)$$

where, we used $\frac{\lambda(\zeta)^{\alpha}}{\Gamma(1+\alpha)}=\frac{(\zeta-x)^{\alpha}}{\Gamma(1+k\alpha)}$ for second-order integro-differential equation as shown in Equation 3.4.

We can use the initial condition to select $u_0(x)=u(0)+\frac{x^{\alpha}}{\Gamma(1+\alpha)}u^{(\alpha)}(0)=1$. Using this selection into the correction functional gives the following successive approximations: Equation 4.11 to 4.15:

$$u_0(x)=1 \quad (4.11)$$

$$u_1(x)=u_0(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_0^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_0(r)(dr)^{\alpha}\right](d\zeta)^{\alpha}$$
$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \quad (4.12)$$

$$u_2(x)=u_1(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_1^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_1(r)(dr)^{\alpha}\right](d\zeta)^{\alpha}$$
$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{8\alpha}}{\Gamma(1+8\alpha)} \quad (4.13)$$

$$u_3(x)=u_2(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_2^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_2(r)(dr)^{\alpha}\right](d\zeta)^{\alpha}$$
$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{8\alpha}}{\Gamma(1+8\alpha)}+\frac{x^{10\alpha}}{\Gamma(1+10\alpha)}+\frac{x^{12\alpha}}{\Gamma(1+12\alpha)} \quad (4.14)$$

And so on:

$$u_n(x)=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\cdots+\frac{x^{4n\alpha}}{\Gamma(1+4n\alpha)}$$

$$u(x)=\lim_{n\to\infty}u_n(x) \tag{4.15}$$

$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{8\alpha}}{\Gamma(1+8\alpha)}+\cdots$$

That gives the exact solution Equation 4.16:

$$u(x)=\cosh_\alpha(x^\alpha). \tag{4.16}$$

Example 3

We consider the local fractional Volterra integro-differential Equation 4.17 and 4.18:

$$u^{(3\alpha)}(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}u(t)(dt)^\alpha\ ,\ u(0)=1,\,u^{(\alpha)}(0)=0,\,u^{(2\alpha)}(0)=1 \tag{4.17}$$

The correction functional for this equation is given by:

$$u_{n+1}(x)=u_n(x)-\frac{1}{\Gamma(1+\alpha)}$$
$$\int_0^x\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}\left[u_n^{(3\alpha)}(\zeta)-1-\frac{\zeta^\alpha}{\Gamma(1+\alpha)}-\frac{\zeta^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+\alpha)}\int_0^\zeta\frac{(\zeta-r)^\alpha}{\Gamma(1+\alpha)}u_n(r)(dr)^\alpha\right](d\zeta)^\alpha \tag{4.18}$$

where, we used $\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)}=-\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}$ for third-order integro-differential equation as shown in (3.4).

Now, we have Equation 4.19 to 4.22:

$$u_0(x)=u(0)+\frac{x^\alpha}{\Gamma(1+\alpha)}u^{(\alpha)}(0)+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}u^{(2\alpha)}(0)=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \tag{4.19}$$

$$u_1(x)=u_0(x)$$
$$-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}\left[u_0^{(3\alpha)}(\zeta)-1-\frac{\zeta^\alpha}{\Gamma(1+\alpha)}-\frac{\zeta^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+\alpha)}\int_0^\zeta\frac{(\zeta-r)^\alpha}{\Gamma(1+\alpha)}u_0(r)(dr)^\alpha\right](d\zeta)^\alpha \tag{4.20}$$
$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{7\alpha}}{\Gamma(1+7\alpha)}$$

$$u_2(x)=u_1(x)$$
$$-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}\left[u_1^{(3\alpha)}(\zeta)-1-\frac{\zeta^\alpha}{\Gamma(1+\alpha)}-\frac{\zeta^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+\alpha)}\int_0^\zeta\frac{(\zeta-r)^\alpha}{\Gamma(1+\alpha)}u_1(r)(dr)^\alpha\right](d\zeta)^\alpha \tag{4.21}$$
$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{7\alpha}}{\Gamma(1+7\alpha)}+\cdots$$

And so on:

$$u_n(x) = 1 + \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots$$
$$+ \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} - \frac{x^{\alpha}}{\Gamma(1+\alpha)} \quad (4.22)$$
$$u(x) = \lim_{n\to\infty} u_n(x)$$
$$= 1 + \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots - \frac{x^{\alpha}}{\Gamma(1+\alpha)}$$

That gives the exact solution Equation 4.23:

$$u(x) = E_{\alpha}(x^{\alpha}) - \frac{x^{\alpha}}{\Gamma(1+\alpha)}. \quad (4.23)$$

5. CONCLUSION

In this study the Volterra integro-differential equations within the local fractional differential operator had been analyzed using the local fractional variational iteration method. The non-differentiable solutions are obtained. The present method is a powerful tool for solving many integral equations within the local fractional derivatives.

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