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# SEQUENTIAL ESTIMATION OF THE SQUARE OF THE RAYLEIGH PARAMETER

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## ABSTRACT

The problem addressed is that of sequentially estimating the square of the parameter of the Rayleigh distribution, subject to a weighted squared loss plus cost of sampling. We propose a sequential procedure and provide a second-order asymptotic expansion for the incurred regret. It is seen that the asymptotic regret is negative for a range of values of the parameter.

**Keywords:** Anscombe's Theorem, Excess Over the Stopping Boundary, Hölder's Inequality, Regret, Sequential Procedure

## 1. INTRODUCTION

Let $X_1,,\ldots,X_n$ denote independent observations to be taken sequentially up to a predetermined stage n from the Rayleigh distribution with p.d.f:

$$f_\theta(x) = \begin{cases} \dfrac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & \text{if } x > 0 \\ 0 & \text{if not,} \end{cases}$$

where, θ is an unknown positive number. It is desired to estimate $\theta^2$, subject to the loss function considered by (Chow and Yu, 1981; Martinsek, 1988) that is Eqution 1:

$$L_a(w_n, \theta^2) = a^2\theta^{4\beta-4}[w_n - \theta^2]^2 + n, \quad (1)$$

where, a is a known positive number, determined by the cost of estimation relative to the cost of a single observation, $\beta>1$ is a given number and $w_n$ is an appropriate point estimate of $\theta^2$ (defined below). In practice, one might be interested in estimating the population variance. $\sigma^2 = ½(4-\pi)\theta^2$ or the population second moment $\mu_2 = 2\theta^2$. Since both of these parameters are linear functions of $\theta^2$, it suffices to estimate $\theta^2$.

For observed values $x_1>0,\ldots, x_n>0$, of $X_1,, \ldots, X_n$, the log-likelihood function is:

$$l_n(\theta) = \sum_{i=1}^n \ln x_i - 2n\ln\theta - \frac{1}{2\theta^2}\sum_{i=1}^n \ln x_i^2$$

For θ>o. It follows that the maximum likelihood estimator of θ is:

$$\hat{\theta}_n = \sqrt{\frac{1}{2n}\sum_{i=1}^n X_i^2} = \sqrt{\bar{Y}_n}$$

where, $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$ with $Y_i = X_i^2/2$, i = 1,…, n and where the random variables $Y_1,\ldots, Y_n$ are independent with common distribution the Exponential distribution with mean $\mu_Y = \theta^2$ and standard deviation $\sigma_Y = \theta^2$.

The risk incurred by estimating $\theta^2$ with $W_n = \hat{\theta}_n^2 = \bar{Y}_n$ under the loss (1) is:

$$R_a(n) = a^2\theta^{4\beta-4}E[(\bar{Y}_n - \theta^2)^2] + n = \frac{a^2\theta^{4\beta}}{n} + n$$

For any fixed value of $a>0$, this risk is minimized with respect to n by choosing n as the greatest integer less than or equal to $n_a = a\theta^{2\beta} = a\sigma_Y^\beta$; in which case, the minimum risk is Equation 2:

$$R_a^* = R_a(n_a) = 2n_a = 2a\sigma_Y^\beta \quad (2)$$

Since $n_a$ depends on the unknown value of θ, there is no fixed-sample-size procedure that attains the minimum risk $R_a^*$ in practice. Therefore, we propose to use the sequential procedure $(T,\bar{Y}_T)$ which stops the sampling process after observing $Y_1,\ldots,Y_T$ and estimates $\theta^2$ by $W_T = \bar{Y}_T$, where Equation 3:

$$T = \inf\left\{ n \geq m_a : n > a\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2\right)^{\beta/2} \right\} \qquad (3)$$

with $m_a$ being a positive integer. Note that the standard deviation based on $Y_1,\ldots, Y_n$ is used in (2) as the estimator of $\theta^2$, instead of $W_n = \bar{Y}_n$, since $\theta^2$ is also the standard deviation of $Y_1$.

If $m_a$ in (3) is such that $\delta\sqrt{a} \leq m_a = o(a)$ as $a\to\infty$ for some $\delta>0$, then Equation 4:

$$E[\bar{Y}_T] = \mu_Y - \frac{\beta}{a}\sigma_Y^{1-\beta} + o\left(\frac{1}{a}\right) = \theta^2 - \frac{\beta}{a\theta^{2\beta-2}} + o\left(\frac{1}{a}\right) \qquad (4)$$

As a θθ, by Martinsek (1988), since the skewness of $Y_1$ is equal to 2. This shows that $\bar{Y}_T$ is biased for large values of a. Thus, consider the biased-corrected estimator Equation 5:

$$\theta_n^* = \bar{Y}_n + \frac{\beta}{a^{1/\beta}n^{1-1/\beta}} \qquad (5)$$

For $n\geq1$, where $\beta>1$. The regret of the sequential procedure $(T,\theta_T^*)$ is defined as Equation 6:

$$r_a(T,\theta_T^*) = E[L_a(T,\theta_T^*)] - R_a^* \qquad (6)$$

where, $R_a^*$ is as in (2). In this study we provide a second-order asymptotic expansion, as as $a\to\infty$, for $r_a(T,\theta_T^*)$ and show that this regret is asymptotically negative if we choose $0 < \theta < \sqrt[6]{(4\beta - 4)/(3.25\beta + 1)}$.

Starr and Woodroofe (1969) considered the case in which $X_1$, $X_2$, … are i.i.d. Normal random variables and showed that the regret of their procedure is $O(1)$. Then, Woodroofe (1977) showed that the regret is 0.5 + o(1) if $m_a\geq4$. Martinsek (1983) extended Woodroofe's result to the nonparametric case. Tahir (1989) proposed a class of bias-reduction estimators of the mean of the one-parameter exponential family and provided an asymptotic second-order lower bound for the regret. Kim and Han (2009) considered estimation of the scale parameter of the Rayleigh distribution under general progressive censoring. Mousa *et al*. (2005; Prakash, 2013) focused on Bayesian prediction and Bayesian estimation for Rayleigh models.

## 2. ASYMPTOTIC EXPANSION FOR THE REGRET OF THE SEQUENTIAL PROCEDURE

Rewrite the stopping time T in (3) as Equation 7:

$$t = \inf\left\{ n \geq m_a : n\left(\frac{V_n}{n}\right)^{-1/2} > a \right\}, where V_n = \sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2 \qquad (7)$$

And let $U_a = t(V_t/t)^{-1/2}-a$ denote the excess over the stopping boundary. Chang and Hsiung (1979) showed that the excess $U_a$ converges in distribution to a random variable U as $a\to\infty$.

### Lemma 1

Let T be as in (3). Then $\frac{T}{a} \to \sigma_Y^\beta = \theta^{2\beta}$ w.p.1 as $a \to \infty$. Moreover:

$$E[T] = a + \nu - 1.375 + o(1)$$

As $a\to\infty$, where $\nu = E[U]$ is the asymptotic mean of the excess over the boundary.

### Proof

The first assertion follows from Lemma 1 of Chow and Robbins (1985). For the second assertion:

$$\begin{aligned} E[T] &= a + \nu - 0.5 - \frac{3}{8\sigma_Y^4}E\left[\left((Y_1 - \mu_Y)^2 - \sigma_Y^2\right)^2\right] + o(1) \\ &= a + \nu - 0.5 - \frac{3}{8}(\kappa - 1) + o(1) \\ &= a + \nu - 1.375 + o(1) \end{aligned}$$

As $a\to\infty$, by Chang and Hsiung (1979), using the fact that the kurtosis of $Y_1$ is $\kappa = \sigma_Y^{-4}E[(Y_1 - \mu_Y)^4] = 6$.

### Proposition 1

Let $\theta_n^*$ be defined by (5) and let T be defined by (3) with ma being such that $\delta\sqrt{a}\leq m_a = o(a)$as $a\to\infty$ for some $\delta>0$. Then, $E[\theta_T^*] = \theta^2 + o(1/a)$ as $a\to\infty$.

### Proof

For $a>0$ Equation 8:

$$aE[\theta_T^* - \theta^2] = aE[\bar{Y}_T - \theta^2] + \beta E\left[\left(\frac{T}{a}\right)^{-(1-1/\beta)}\right] \tag{8}$$

The proposition follows by taking the limit as $a\to\infty$ in (8) and using (4) and the fact that $E[(T/a)^{-(1-1/\beta)}] \to \sigma_Y^{1-\beta}$ as $a\to\infty$ if $\beta>1$, by the first assertion of Lemma 1 and (2.2) of Martinsek (1983).

Let $r_a(T,\theta_T^*)$ be as in (6). Then Equation 9:

$$\begin{aligned} r_a(T,\theta_T^*) &= E[a^2\sigma_Y^{2\beta-2}(\bar{Y}_T-\mu_Y)^2 + T - 2a\sigma_Y^\beta] \\ &+2\beta\sigma_Y^{2\beta-2}a^{2-1/\beta}E\left[\frac{1}{T^{1-1/\beta}}\left(\bar{Y}_T-\mu_Y\right)\right] \\ &+\beta^2\sigma_Y^{2\beta-2}E\left[\frac{a^{2-2/\beta}}{T^{2-2/\beta}}\right] \\ &= r_a(T,\bar{Y}_T) + 2\beta\sigma_Y^{2\beta-2}E\left[\frac{a^{1-1/\beta}}{T^{1-1/\beta}}a\left(\bar{Y}_T-\mu_Y\right)\right] \\ &+\beta^2\sigma_Y^{2\beta-2}E\left[\frac{a^{2-2/\beta}}{T^{2-2/\beta}}\right] \end{aligned} \tag{9}$$

## Lemma 2

Let T be defined by (3) with $m_a$ being such that $\delta\sqrt{a}\le ma = o(a)$ as $a\to\infty$ for some $\delta>0$ and with $\beta>1$. Then:

$$E\left[\frac{a^{1-1/\beta}}{T^{1-1/\beta}}a\left(\bar{Y}_T-\mu_Y\right)\right] = \frac{2(1-\beta)}{\sigma_Y^{2\beta+1}} - \frac{\beta}{\sigma_Y^{2\beta-2}} + o(1)$$

As $a\to\infty$.

## Proof

First, observe that Equation 10:

$$\begin{aligned} &E\left[\frac{a^{1-1/\beta}}{T^{1-1/\beta}}a\left(\bar{Y}_T-\mu_Y\right)\right] \\ &= E\left[\left(\left(\frac{a}{T}\right)^{1-1/\beta} - \frac{1}{\sigma_Y^{\beta-1}}\right)a\left(\bar{Y}_T-\mu_Y\right)\right] + \frac{1}{\sigma_Y^{\beta-1}}aE[\bar{Y}_T-\mu_Y] \end{aligned} \tag{10}$$

For $a>0$. Moreover Equation 11:

$$aE[\bar{Y}_T-\mu_Y] = -\frac{\beta}{\sigma_Y^{\beta-1}} + o(1) \tag{11}$$

As $a\to\infty$, by (4). Next, expand $g(y) = y^{1/\theta 1}$ at $y=\sigma_Y^\beta$, substitute $y = T/a$ and multiply by $a\left(\vec{Y}_T - \mu_Y\right)$ to obtain Equation 12:

$$\begin{aligned} &\left(\frac{a^{1-1/\beta}}{T^{1-1/\beta}} - \frac{1}{\sigma_Y^{\beta-1}}\right)(\bar{Y}_T-\mu_Y) \\ &= \left(\frac{1}{\beta}-1\right)T_*^{1/\beta-2}\left(\frac{T}{a}-\sigma_Y^\beta\right)a(\bar{Y}_T-\mu_Y) \end{aligned} \tag{12}$$

where, $T_*$ is a random variable such that $\left|T_* \sigma_Y^\beta\right| \le \left|T/a - \sigma_Y^\beta\right|$. Next, rewrite T in (3) as $T = \inf\{n \ge m_a: n(V_n/n)^{-\beta/2} > a\}$, where, $V_n$ is as in (7) and let:

$$U_a^* = T\left(\frac{V_T}{T}\right)^{-\beta/2} - a$$

Denote the excess over the stopping boundary. Expanding $h(y) = y^{-\beta/2}$ at $y=\sigma_Y^2$, substituting $y = V_T/T$ and multiplying by T yields:

$$\begin{aligned} T\left(\frac{V_T}{T}\right)^{-\beta/2} &= \frac{T}{\sigma_Y^\beta} - \frac{\beta}{2\sigma_Y^{\beta+2}} \\ &(V_T - T\sigma_Y^2) + \frac{\beta(\beta+2)}{8\lambda_T^{\beta/2+2}}\frac{(V_T-T\sigma_Y^2)^2}{T} \end{aligned}$$

for $a>0$, where $\lambda_T$ is a random variable between $V_T/T$ and $\sigma_Y^2$. Furthermore, write:

$$V_T = \sum_{i=1}^{T}(Y_i - \mu_Y)^2 - T(\bar{Y}_T - \mu_Y)^2$$

To obtain:

$$\begin{aligned} U_a^* &= \frac{T}{\sigma_Y^\beta} - a - \frac{\beta}{2\sigma_Y^{\beta+2}}(W_T - T\sigma_Y^2) \\ &+\frac{\beta}{2\sigma_Y^{\beta+2}}T(\bar{Y}_T-\mu_Y)^2 + \frac{\beta(\beta+2)}{8\lambda_T^{\beta/2+2}}\frac{(V_T-T\sigma_Y^2)^2}{T} \end{aligned}$$

For $a>0$, where $W_T = \sum_{i=1}^{T}(Y_i-\mu_Y)^2$. It follows easily that Equation 13:

$$\frac{T}{a} - \sigma_Y^\beta = \frac{\sigma_Y^\beta}{a}(U_a^* - \xi_T) + \frac{\beta}{2a\sigma_Y^2}(W_T - T\sigma_Y^2) \tag{13}$$

For $a>0$, where:

$$\xi_T = \frac{\beta}{2\sigma_Y^{\beta+2}}T(\bar{Y}_T-\mu_Y)^2 + \frac{\beta(\beta+2)}{8\lambda_T^{\beta/2+2}}\frac{(V_T-T\sigma_Y^2)^2}{T}$$

Substituting (13) in (12) yields Equation 14:

$$\left(\frac{a^{1-1/\beta}}{T^{1-1/\beta}} - \frac{1}{\sigma_Y^{\beta-1}}\right)(\bar{Y}_T - \mu_Y)$$
$$= \left(\frac{1}{\beta} - 1\right)\sigma_Y^{\beta} T_*^{1/\beta-2}(U_a - \xi_T)(\bar{Y}_T - \mu_Y)$$
$$+ \left(\frac{1}{\beta} - 1\right)\frac{\beta}{2\sigma_Y^2} T_*^{1/\beta-2}(W_T - T\sigma_Y^2)(\bar{Y}_T - \mu_Y) \quad (14)$$
$$= \left(\frac{1}{\beta} - 1\right)\sigma_Y^{\beta} I_1(a) + \frac{1-\beta}{2\sigma_Y^2} I_2(a),$$

Say. Let $S_n = Y_1+\ldots+Y_n$, n≥1. Then Equation 15:

$$E[|I_1(a)|] = E\left[\left|\frac{T_*^{1/\beta-2}}{T}(U_a - \xi_T)(S_T - \mu_Y T)\right|\right]$$
$$= \frac{\sigma_Y^{\beta}}{\sqrt{a\sigma_Y^{\beta}}} E\left[\left|(U_a - \xi_T)\frac{a}{T} T_*^{1/\beta-2}\frac{(S_T - \mu_Y T)}{\sqrt{a\sigma_Y^{\beta}}}\right|\right]$$
$$\le \frac{\sqrt{\sigma_Y^{\beta}}}{\sqrt{a}}\sqrt{E[(U_a - \xi_T)^2]}\sqrt{E\left[T_*^{2/\beta-4}\left(\frac{a}{T}\right)^2\left(\frac{S_T - \mu_Y T}{\sqrt{a\sigma_Y^{\beta}}}\right)^2\right]} \quad (15)$$
$$\le \frac{1}{\sqrt{a}}\sqrt{2\sigma_Y^{\beta} E[U_a^2] + 2\sigma_Y^{\beta} E[\xi_T^2]}$$
$$\sqrt{E\left[T_*^{2/\beta-4}\left(\frac{a}{T}\right)^2\left(\frac{S_T - \mu_Y T}{\sqrt{a\sigma_Y^{\beta}}}\right)^2\right]} \to 0$$

as a→∞, by Hölder's inequality, the fact that $T_* \to \sigma_Y^{\beta}\left(\left|T_* - \sigma_Y^{\beta}\right| \le \left|T/a - \sigma_Y^{\beta}\right| \to 0 \text{ w.p.1}\right.$ since $T/a \to \sigma_Y^{\beta}$, $\frac{S_T - \mu_Y T}{\sqrt{a\sigma_Y^{\beta}}}$ converges in distribution to a Standard Normal random variable by Anscombe's theorem, the facts that $E[U_a^2] \to E[U^2] < \infty$ and $E[\xi_T^2] = O(1)$ $a\to\infty$ and (2.3), (2.8) and (2.9) of Martinsek (1983). To evaluate $E[I_2(a)]$, observe that Equation 16:

$$I_2(a) = \frac{2a\sigma_Y^{\beta}}{T} T_*^{1/\beta-2}\frac{(W_T - T\sigma_Y^2)(S_T - \mu_Y T)}{a\sigma_Y^{\beta}}$$
$$= 2\sigma_Y^{\beta}\frac{a}{T} T_*^{1/\beta-2}\left(\frac{W_T - \sigma_Y^2 T}{\sqrt{a\sigma_Y^{\beta}}} + \frac{S_T - \mu_Y T}{\sqrt{a\sigma_Y^{\beta}}}\right)^2$$
$$-2\sigma_Y^{\beta}\frac{a}{T} T_*^{1/\beta-2}\left(\frac{W_T - \sigma_Y^2 T}{\sqrt{a\sigma_Y^{\beta}}}\right)^2 \quad (16)$$
$$-2\sigma_Y^{\beta}\frac{a}{T} T_*^{1/\beta-2}\left(\frac{S_T - \mu_Y T}{\sqrt{a\sigma_Y^{\beta}}}\right)^2$$
$$\xrightarrow{\text{in distribution}} 2\sigma_Y^{1-2\beta}(2Z)^2$$
$$-2\sigma_Y^{1-2\beta} Z^2 - 2\sigma_Y^{1-2\beta} Z^2 = 4\sigma_Y^{1-2\beta} Z^2$$

As a→∞, by Anscombe's theorem and the fact that $T_* ® \sigma_Y^{\beta}$ w.p.1 as a→∞ where Z is a random variable having the Standard Normal distribution. Thus Equation 17:

$$E[I_2(a)] = 4\sigma_Y^{1-2\beta} + o(1) \quad (17)$$

As $a$→∞, by (16) and (2.3) and (2.4) of Martinsek (1983). Taking expectation in (14) and using (15) and (17) yields Equation 18:

$$E\left[\left(\frac{a^{1-1/\beta}}{T^{1-1/\beta}} - \frac{1}{\sigma_Y^{\beta-1}}\right)a\left(\bar{Y}_T - \mu_Y\right)\right] = \frac{2(1-\beta)}{\sigma_Y^{2\beta+1}} + o(1) \quad (18)$$

$a$→∞. The lemma follows by taking the limit, as $a$→∞, in (10) and using (11) and (18).

Theorem 1. Let T be defined by (3) with $m_a$ being such that $\delta\sqrt{a} \le m_a = o(a)$ as $a$→∞ for some δ>0 and β>1. Let the regret of the biased-corrected procedure $(T,\theta_T^*)$ be as in (6). Then:

$$r_a(T,\theta_T^*) = 3.25\beta^2 + \beta - \frac{4\beta(\beta-1)}{\theta^6} + o(1)$$

As αθθ.

### Proof

First Equation 19:

$$r_a(T,\bar{Y}_T) = E[a^2\sigma_Y^{2\beta-2}(\bar{Y}_T - \mu_Y)^2 + T]$$
$$-2a\sigma_Y^{\beta} = 5.25\beta^2 + \beta + o(1) \quad (19)$$

As a→∞ if δ>1, by Martinsek (1988). Next, take the limit, as a→∞, in (9) and use (19), Lemma 2 and the fact that:

$$E\left[\frac{a^{2-2/\beta}}{T^{2-2/\beta}}\right] = \frac{1}{\sigma_Y^{2\beta-2}} + o(1)$$

as a→∞ if δ>1, by the first assertion of Lemma 1 and (2.2) of Martinsek (1983), to complete the proof.

## 3. NEGATIVE ASYMPTOTIC REGRET

Theorem 1 shows that the biased-corrected procedure $(T,\mu_T^*)$ has a lower asymptotic regret than the procedure $(T,\bar{Y}_T)$. Also, the asymptotic regret of the procedure $(T,\mu_T^*)$ is negative if Equation 20:

$$0 < \theta < \sqrt[6]{\frac{4\beta - 4}{3.25\beta + 1}} \equiv \theta_\beta \quad (20)$$

**Table 1.** Asymptotic regret for various choices of $\beta>1$ and $0<\theta<\theta_\beta$ (see (20))

| b | $q_b$ | q | Asymptotic regret |
|---|---|---|---|
| 1.5 | 0.836 | 0.2 | -46866.1880000 |
| 1.5 | 0.836 | 0.3 | -4106.4138000 |
| 1.5 | 0.836 | 0.4 | -723.6093800 |
| 1.5 | 0.836 | 0.5 | -183.1875000 |
| 1.5 | 0.836 | 0.7 | -16.6870790 |
| 2.0 | 0.901 | 0.2 | -124985.0000000 |
| 2.0 | 0.901 | 0.3 | -10958.9370000 |
| 2.0 | 0.901 | 0.4 | -1938.1250000 |
| 2.0 | 0.901 | 0.5 | -497.0000000 |
| 2.0 | 0.901 | 0.8 | -15.5175780 |
| 2.0 | 0.901 | 0.9 | -0.0534114 |
| 5.0 | 0.988 | 0.3 | -109653.12.00000 |
| 5.0 | 0.988 | 0.4 | -19445.0000000 |
| 5.0 | 0.988 | 0.6 | -1628.4276000 |
| 5.0 | 0.988 | 0.7 | -593.7387800 |
| 5.0 | 0.988 | 0.9 | -64.2841140 |
| 10 | 1.012 | 0.2 | -5624665.0000000 |
| 10 | 1.012 | 0.4 | -87555.6250000 |
| 10 | 1.012 | 0.7 | -2724.9495000 |
| 10 | 1.012 | 0.8 | -1038.2910000 |
| 10 | 1.012 | 0.9 | -342.4035100 |
| 10 | 1.012 | 1.0 | -25.0000000 |
| 15 | 1.020 | 0.2 | -13124254.0000000 |
| 15 | 1.020 | 0.3 | -1151517.1000000 |
| 15 | 1.020 | 0.5 | -53013.7500000 |
| 15 | 1.020 | 0.7 | -6393.6322000 |
| 15 | 1.020 | 0.9 | -834.3582000 |
| 15 | 1.020 | 1.0 | -93.7500000 |

This means that for the values of $\theta$ in the interval $(0, \theta_\beta)$ with $\beta>1$, the sequential procedure $(T,\mu_T^*)$ performs better, for large values of a, than the best fixed-sample-size procedure $(n_a^*,\bar{Y}_{n_a^*})$, where $n_a^*$ is the greatest integer less than or equal to $n_a = a\theta^{2\beta}$ (see **Table 1**).

## 4. CONCLUSION

We have proposed a sequential procedure for estimating the square of the shape parameter of the Rayleigh distribution and provided a second-order asymptotic expansion for the incurred regret. It is seen that the proposed procedure performs better than the best fixed-sample-size procedure if the shape parameter lies in a specific subinterval of the positive real numbers.

For future research, it would be worth considering Bayesian sequential estimation of a function of the shape parameter of the Rayleigh distribution, in which the focus will be on finding a sequential procedure and approximating the Bayes regret, as well as comparing the proposed procedure with existing procedures.

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