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CHARACTERIZATION OF MARKOV-BERNOULLI GEOMETRIC DISTRIBUTION RELATED TO RANDOM SUMS

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ABSTRACT

The Markov-Bernoulli geometric distribution is obtained when a generalization, as a Markov process, of the independent Bernoulli sequence of random variables is introduced by considering the success probability changes with respect to the Markov chain. The resulting model is called the Markov-Bernoulli model and it has a wide variety of application fields. In this study, some characterizations are given concerning the Markov-Bernoulli geometric distribution as the distribution of the summation index of independent randomly truncated non-negative integer valued random variables. The achieved results generalize the corresponding characterizations concerning the usual geometric distribution.

Keywords: Markov-Bernoulli Geometric Distribution, Random Sum, Random Truncation, Characterization

1. INTRODUCTION

The Markov-Bernoulli Geometric (MBG) distribution has been obtained by Anis and Gharib (1982) in a study of Markov-Bernoulli sequence of random variables (rv's) introduced by Edwards (1960), who generalized the (usual) independent Bernoulli sequence of rv's by considering the success probability changes with respect to a Markov chain. The resulting model is called the Markov-Bernoulli Model (MBM) or the Markov modulated Bernoulli process (Ozekici, 1997). Many researchers have been studied the MBM from the various aspects of probability, statistics and their applications, in particular the classical problems related to the usual Bernoulli model (Anis and Gharib, 1982; Gharib and Yehia, 1987; Inal, 1987; Yehia and Gharib, 1993; Ozekici, 1997; Ozekici and Soyer, 2003; Arvidsson and Francke, 2007; Omey *et al*., 2008; Maillart *et al*., 2008; Pacheco *et al*., 2009; Cekanavicius and Vellaisamy, 2010; Minkova and Omey, 2011). Further, due to the fact that the MBM operates in a random environment depicted by a Markov chain so that the probability of success at each trial depends on the state of the environment, this model represents an interesting application of stochastic processes and thus used by numerous authors in, stochastic modeling (Switzer, 1967; 1969; Pedler, 1980; Satheesh *et al*., 2002; Özekici and Soyer, 2003; Xekalaki and Panaretos, 2004; Arvidsson and Francke, 2007; Nan *et al*., 2008; Pacheco *et al*., 2009; Doubleday and Esunge, 2011; Pires and Diniz, 2012).

Let X_1, X_2,… be a sequence of Markov-Bernoulli rv's with the following matrix of transition probabilities Equation (1.1):

$$\begin{array}{cc} & X_{i+1} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ X_i \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-(1-\rho)p & (1-\rho)p \\ (1-\rho)(1-p) & \rho+(1-\rho)p \end{bmatrix}, \end{array} \quad (1.1)$$

and initial distribution:

$$P(X_1=1)=p=1-P(X_1=0)$$

where, $0 \le p \le 1$ and $0 \le \rho \le 1$.

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The sequence $\{X_i\}$ with the transition matrix (1.1) and the above initial distribution is called the MBM. If E_i, i = 0, 1 are the states of the Markov system given by (1.1), then the parameter ρ which is usually called the persistence indicator of E_0, is the correlation coefficient between X_i and X_{i+1}, $i = 1, 2,\ldots$ (Anis and Gharib, 1982).

If N is the number of transitions for the system defined by (1.1) to be in E_1 for the first time then N has the MBG distribution given by Equation (1.2):

$$P(N=k)=p_k=\begin{cases}1-\alpha, & k=0\\ \alpha(1-t)t^{k-1}, & k\geq 1,\end{cases} \quad (1.2)$$

where:

$\alpha = 1\text{-}p$ and $t = \rho + (1-\rho)\alpha = \alpha + (1-\alpha)\rho$.

The MBG distribution (1.2) will be denoted by MBG(α, ρ) and we shall write N~MBG(α, ρ). Let $P_U(s) = E(s^U)$ be the probability generating function (pgf) of an integer valued random variable U, $|s|\leq 1$.

The pgf of N is given, using (1.2), by Equation (1.3):

$$P_N(s)=\sum_{k=0}^{\infty} p_k s^k = \frac{(1-\alpha)(1-\rho s)}{(1-ts)}. \quad (1.3)$$

Few authors have treated the characterization problem of the MBG distribution (Yehia and Gharib, 1993; Minkova and Omey, 2011).

In this study some new characterizations are given for the MBG distribution by considering it as the distribution of the summation index of a random sum of randomly truncated non-negative integer valued rv's. The scheme of geometric random sum of randomly truncated rv's is important in reliability theory, especially in the problem of optimal total processing time with checkpoints (Dimitrov *et al*., 1991). The achieved results generalize those given in (Khalil *et al*., 1991).

2. RANDOM SUM OF TRUNCATED RANDOM VARIABLES

Consider a sequence $\{X_n, n\geq 1\}$ of independent identically distributed (iid) non-negative integer valued rv's with probability mass function (pmf) $0 \leq p_k = P(X_1 = k)$; $k = 0, 1, 2, \ldots$.

Let $\{Y_n, n\geq 1\}$ be another sequence of iid non-negative integer valued random variables, independent of $\{X_n\}$, with pmf $q_k = P(Y_1 = k)$; $k = 0, 1, 2,\ldots$, $\sum_{k=0}^{\infty} q_k = 1$. $\{Y_n\}$ is called the truncating process.

Put $p = P(X_1\geq Y)$ and assume $0< p <1$. Let $N = \inf\{k \geq 1: X_k<Y_k\}$. Clearly,

$P(N\text{-}1= k) = p^k (1\text{-}p)$, $k = 0,1,2,\ldots$, i.e., N-1 has a geometric distribution with parameter p.

Remark 1

If the sequence $\{X_n, n\geq 1\}$ follows the MBM (1.1), where the event $\{(X_i \geq Y_i)\}$ indicates the state E_0 and the event $\{(X_i<Y_i)\}$ indicates the state E_1, with the initial distribution:

$$P(X_1 \geq Y_1) = p \text{ and } P(X_1 < Y_1) = 1-p$$

Then the rv N (the smallest number of transitions for the system to be in state E_1 for the first time) wouldhave the MBG distribution defined by (1.2) (Anis and Gharib, 1982) Equation (2.1):

Define:

$$Z=\sum_{k=0}^{N-1} Y_k + X_N \quad (2.1)$$

where, for convenience $Y_0 = 0$. The rv Z is the random sum of truncated rv's (Khalil *et al*., 1991).

It is known that the pgf $P_Z(s)$ of Z is given by (Khalil *et al*., 1991) Equation (2.2 to 2.5):

$$P_Z(s)=P_1(s)/\left[1-Q_1(s)\right], \quad (2.2)$$

where:

$$P_1(s)=E\left[S^{X_1} I(X_1<Y_1)\right]=\sum_{k=0}^{\infty} p_k s^k \sum_{\gamma=k+1}^{\infty} q_\gamma, \quad (2.3)$$

$$Q_1(s)=E\left[S^{Y_1} I(X_1\geq Y_1)\right]=\sum\nolimits_{k=0}^{\infty} q_k s^k \sum\nolimits_{\gamma=k}^{\infty} p_\gamma, \quad (2.4)$$

and $I(A)$ is the indicator function of the set A.

Corollary 1

$$E(Z)=\frac{\left(\sum_{k=0}^{\infty} kp_k \sum_{\gamma=k+1}^{\infty} q_\gamma\right)}{\left(1-\sum_{k=0}^{\infty} kq_k \sum_{\gamma=k+1}^{\infty} p_\gamma\right)} \quad (2.5)$$

The result is immediate since $E(Z) = P'z(1)$.

Corollary 2

If the truncating process $\{Y_n\}$ is such that Y_n~MBG(α, ρ), for $\alpha\in(0,1)$ and $\rho\in[0, 1]$. Then the pgf of Z is given by Equation (2.6 and 2.7):

$$P_Z(s)=\frac{(1-ts)P_{X_1}(ts)}{\left[1-\left\{1-(1-t)P_{X_1}(ts)\right\}s\right]}$$
$$=\frac{\alpha P_{X_1}(ts)}{\left[1-\frac{(1-\alpha)}{(1-ts)}\left\{1-s\left(1-\frac{(1-t)}{(1-\alpha)}\left[1-\alpha P_{X_1}(ts)\right]\right)\right\}\right]}, \quad (2.6)$$

and:

$$E(Z)=P_Z'(1)=\frac{\left[1-P_{X_1}(t)\right]}{\left[(1-t)P_{X_1}(t)\right]}, \quad (2.7)$$

where, $t = \rho+(1-\rho)\alpha$.

Proof

Since Y_n~MBG(α, ρ), then:

$$P(Y_n=k)=q_k=\begin{cases}1-\alpha, & k=0\\ \alpha(1-t)t^{k-1}, & k=1,2,\ldots\end{cases}$$

Hence, using (2.3), we have Equation 2.8 and 2.9:

$$P_1(s)=\sum_{k=0}^{\infty}p_k s^k\sum_{\gamma=k+1}^{\infty}q_\gamma$$
$$=p_0\sum_{\gamma=1}^{\infty}q_\gamma+\sum_{k=1}^{\infty}p_k s^k\sum_{\gamma=k+1}^{\infty}q_\gamma=\alpha P_{X_1}(ts), \quad (2.8)$$

and, using (2.4), we have:

$$Q_1(s)=q_0\sum_{\gamma=0}^{\infty}p_\gamma+\sum_{k=1}^{\infty}q_k s^k\sum_{\gamma=k}^{\infty}p_\gamma$$
$$=(1-\alpha)+\alpha(1-t)s\sum_{\gamma=1}^{\infty}p_\gamma\sum_{k=1}^{\gamma}(ts)^{k-1} \quad (2.9)$$
$$=\frac{(1-\alpha)}{(1-ts)}\left[1-s\left\{\rho+\alpha(1-\rho)P_{X_1}(ts)\right\}\right].$$

Substituting the expressions of $P_1(s)$ and $Q_1(s)$ given respectively by (2.8) and (2.9) into (2.2), we get (2.6).

$E(Z)$ is obtained readily from (2.5) or (2.6).

3. CHARACTERIZATIONS OF THE MBG DISTRIBUTION

Consider the random sum given by (2.1).

Theorem 1

Let Y_1, have a MBG distribution with some parameters $\alpha\in(0, 1)$ and $\rho\in[0, 1]$ and let X_1 satisfy $P(X_1 = 0)<1$. Then Z has the same distribution as X_1 ($Z\overset{d}{=}X_1$) if and only if X_1 has a MBG distribution.

Proof

Assume that X_1~MBG(β, ρ_1), for some $\beta\in(0, 1)$ and $\rho_1\in[0, 1]$,i.e.,:

$$p_k=\begin{cases}1-\beta, & k=0\\ \beta(1-\ell)\ell^{k-1}, & k=1,2,\ldots,\end{cases}$$

where, $\ell = \rho_1+(1-\rho_1)\beta$. Then the pgf of X_1 is given by Equation (3.1 and 3.2):

$$P_{X_1}(s)=\frac{(1-\beta)(1-\rho_1 s)}{(1-\ell s)}. \quad (3.1)$$

Substituting, (3.1) into (2.6), we obtain:

$$P_Z(s)=\frac{(1-\beta)(1-ts)(1-\rho_1 ts)}{1-s\left[1+\ell t-\ell ts-(1-t)(1-\beta)(1-\rho_1 ts)\right]}$$
$$=\frac{(1-\beta)(1-ts)(1-\rho_1 ts)}{ts^2\left[\beta+\rho_1 t(1-\beta)\right]-s\left[\beta+\rho_1 t(1-\beta)+t\right]+1} \quad (3.2)$$
$$=\frac{(1-\beta)(1-\rho_2 s)}{(1-Ds)},$$

where, $\rho_2 = \rho_1$t and $D = \beta+\rho_2(1-\beta)$.

It follows from (3.2), that the rv Z has a MBG distribution with parameters $\beta\in(0, 1)$ and $\rho_2\in[0, 1]$.

Conversely, assume that $P_Z(s) = P_{X_1}(s)$ where Y_1~ MBG(α, ρ). Hence, we have to show that $P_{X_1}(s)$ is the pgf of MBG distribution. Replacing $P_Z(s)$ by $P_{X_1}(s)$ in (2.6), yields in: $(1-ts)P_{X_1}(ts[1-s\{1-(1-t)P_{X_1}(ts)\}] = P_{X_1}(s)$. Or: $(1-ts)P_{X_1}(ts) = (1-s)P_{X_1}(s)+s(1-t)P_{X_1}(s)P_{X_1}(ts)$.

Dividing both sides of this equation by $(1-s)(1-ts)P_{X_1}(s)P_{X_1}(ts)$, we get:

$$\frac{1}{(1-ts)P_{X_1}(ts)}+\frac{s(1-t)}{(1-s)(1-ts)}=\frac{1}{(1-s)P_{X_1}(s)}.$$

Now, using partial fractions for the middle term and using some manipulations, we can write Equation (3.3):

$$\frac{P_{X_1}^{-1}(ts)-1}{(1-ts)}=\frac{P_{X_1}^{-1}(s)-1}{(1-s)} \quad (3.3)$$

Or, $H(ts) = H(s)$, where, $H(s)=\frac{P_{X_1}^{-1}(s)-1}{(1-s)}$.

Putting $s = 1$, we get:

$$H(t)=H(1)=lim_{s\to 1}\frac{P_{X_1}^{-1}(s)-1}{(1-s)}=E(X_1)=C,\text{say}$$

Therefore, the solution of (3.3) is given by:

$$P_{X_1}(s)=\left[1+C(1-s)\right]^{-1}.$$

Choosing $C = \beta/(1-\beta)\{1-[\alpha+(1-\alpha)\rho]\rho_1\}$and recalling that $\alpha + (1-\alpha)\rho = t$, we finally get:

$$P_{X_1}(s)=\frac{(1-\beta)(1-\rho_1 s)}{(1-\ell s)},$$

where, $\ell = \rho_1+(1-\rho_1)\beta$.

Which is the pgf of the MBG(β, ρ_1) distribution. Hence X_1~MBG(β, ρ_1).
This completes the proof of Theorem 1.

Remark 2

It follows from Theorem 1 that when X_1 has the MBG distribution then the random sum Z and the summands have distributions of the same type and in this case the summands are called N-sum stable (Satheesh *et al*., 2002). This result is valid, also, as a consequence of the fact that geometric random sums are stable in the same sense.

Another characterization for the MBG distribution can be obtained in terms of the expected values of Z and X_1.

Theorem 2

Let Y_1 have a MBG distribution with parameters $\alpha\in(0,1)$ and $\rho\in[0,1]$ and consider $A =\{(\alpha, \rho): \alpha\in (0, 1), \rho\in [0, 1]\}$. Then X_1 has a MBG(β, ρ_1) distribution if and only if $\forall(\alpha,\rho)\in A$, $E(Z) =\beta/ (1-\beta)(1-\rho_2)]$, where $\rho_2 = \rho_1 t$ and $t = \rho + (1 -\rho)\alpha$.

Proof

If X_1~MBG(β, ρ_1) for some $(\beta, \rho_1) \in A$, then it follows from Theorem 1 that Z~MBG($\hat{a}$, ρ_2), where $\rho_2 = \rho_1 t$ and $t = \rho + (1 -\rho)\alpha$. Consequently, $E(Z) = \beta/[(1-\beta)(1-\rho_2)]$.

Conversely, suppose that: $E(Z) = \beta/[(1-\beta)(1-\rho_2)]$ for some$(\beta, \rho_1) \in A$.

Then from (2.7), we have:

$$\frac{\beta}{(1-\beta)(1-\rho_2)}=\frac{1-P_{X_1}(t)}{(1-t)P_{X_1}(t)}.$$

Solving this equation with respect to $P_{X_1}(s)$, one has:

$$P_{X_1}(t)=\frac{(1-\beta)(1-\rho_1 t)}{(1-\ell t)},\ \forall(\alpha,\beta)\in A.$$

Which is the pgf of the MBG (β, ρ_1) distribution.
Hence X_1~MBG(β, ρ_1).
This completes the proof of Theorem 2.

The following theorem expresses the relation between the distribution of Z and the distribution of the truncating process $\{Y_n\}$.

Theorem 3

Let X_1 have a MBG(β,ρ_1) distribution for some parameters $\beta\in(0,1)$ and $\rho_1\in[0,1]$ and let Y_1 satisfy $q_0 = P(Y_1 = 0) < 1$. Then $Z\overset{d}{=}X_1$ if and only if Y_1 follows a MBG distribution.

Proof

Since X_1~MBG(β, ρ_1), for some $\beta\in(0, 1)$ and $\rho\in[0, 1]$, then:

$$p_k=\begin{cases}1-\beta, & k=0\\ \beta(1-\ell)\ell^{k-1} & k=1,2,...,\end{cases}$$

where, $\ell = \rho_1+(1-\rho_1)\beta$.

Proceeding as in Corollary 2, one has:

$$\begin{aligned}P_1(s)&=\sum_{k=0}^{\infty}p_k s^k\sum_{\gamma=k+1}^{\infty}q_\gamma\\&=(1-\beta)(1-q_0)+\beta(1-\ell)s\sum_{\gamma=2}^{\infty}q_\gamma\sum_{k=1}^{\gamma-1}(\ell s)^{k-1} \qquad (3.4)\\&=\frac{(1-\beta)}{\ell(1-\ell s)}[\ell-\rho_1 q_0-\ell s\rho_1(1-q_0)-\beta(1-\rho_1)P_{Y_1}(\ell s)],\end{aligned}$$

and:

$$Q_1(s) = \sum_{k=0}^{\infty} p_k s^k \sum_{\gamma=k}^{\infty} p_\gamma$$
$$= q_0 + \frac{\beta}{\ell} \sum_{k=1}^{\infty} q_k (\ell s)^k = \frac{1}{\ell}\left[\beta P_{Y_1}(\ell s) + q_0 \rho_{Y_1}(1-\beta)\right] \quad (3.5)$$

Substituting (3.4) and (3.5) into (2.2), we obtain:

$$P_Z(s) = \frac{(1-\beta)}{(1-\ell s)}\left[\frac{\ell - \rho_1 q_0 - \ell\rho_1(1-q_0)s - \beta(1-\rho_1)P_{Y_1}(\ell s)}{\ell - \rho_1 q_0(1-\beta) - \beta P_{Y_1}(\ell s)}\right]$$
$$= \frac{(1-\beta)}{(1-\ell s)}\left[\frac{\rho_1 + (1-\rho_1)\beta - \rho_1 q_0 - \ell\rho_1(1-q_0)s - \beta(1-\rho_1)P_{Y_1}(\ell s)}{\rho_1(1-\beta)(1-q_0) + \beta\left(1 - P_{Y_1}(\ell s)\right)}\right] \quad (3.6)$$

Now assume that $Z \overset{d}{=} X_1$, then $P_Z(s) = P_{X_1}(s)$.

Consequently, equating (3.1) and (3.6) and then solving for $P_{Y_1}(\ell s)$, we get:

$$P_{Y_1}(\ell s) = \frac{\beta q_0 + \left[\rho_1(1-q_0) - \beta\right]s}{\beta(1-s)}$$

From which we finally have:

$$P_{Y_1}(s) = \frac{q_0\left\{1 - (\rho q_0)^{-1}\left[1 - \rho_1\beta^{-1}(1-q_0)\right]s\right\}}{\left(1 - \ell^{-1}s\right)}.$$

Which is the pgf of a MBG(γ, ρ_2) distribution with, $\gamma = 1 - q_0$ and $\rho_2 = (\rho q_0)\,[1 - \rho_1\beta^{-1}(1 - q_0)]$.

Hence Y_1~MBG(γ, ρ_2).

The "only if" part of the proof follows directly by applying Theorem 1.

This completes the proof of Theorem 3.

Remark 3

The results of (Khalil *et al*., 1991) follow as special cases from our corresponding results when the MBM (1.1) reduces to the independence case by putting the correlation parameter $\rho = 0$.

4. CONCLUSION

In this study three characterizations for the Markov-Bernoulli geometric distribution are proved. These results extend the corresponding characterizations of the geometric distribution. Further, the achieved results have a direct relevance to the stability problem of random sums of random variables. Moreover, the given characterization theorems will be useful, in understanding the missing link between the mathematical structure of Markov-Bernoulli geometric distribution and the actual behavior of some real world random phenomena.

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