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EXISTENCE OF SOLUTIONS FOR A CLASS OF VARIATIONAL INEQUALITIES

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ABSTRACT

In this study we considered a deformed elastic solid with a unilateral contact of a rigid body. We studied the existence, uniqueness and continuity of the deformation of this solid with respect to the data. We proved the existence of solutions for a class of variational inequalities.

Keywords: Variational Inequalities, Unilateral Contact

1. INTRODUCTION

Several problems in mechanics, physics, control and those dealing with contacts, lead to the study of systems of variational inequalities.

This model has been studied by Slimane *et al*. (2004); Bernardi *et al*. (2004); Brezis (1983); Brezis and Stampacchia (1968); Ciarlet (1978); Grisvard (1985); Haslinger *et al*. (1996); Lions and Stampacchia (1967).

We consider a solid occupying an open bounded domain Ω of a sufficiently regular boundary $\Gamma = \partial\Omega$ with unilateral contact with a rigid obstacle.

Theorem 1.1

Let $P \in L^2$ (ΩR^3) be the resulting of force density. Then there exists a unique solution for the variational problem: find $u \in V$ such that:

$$a(u,v) = l(v), \forall v \in V$$

With:

$$V = \left\{ v \in H_0^1(\Omega, R^3) \right\}$$

a(u,v) = The bilinear form
l(v) = The linear form

To prove this theorem we make use of the Lax-Milgram which is based on proving the continuity and V-ellipticity of the bilinear form a(u,v) and the continuity of l(v).

2. FORMULATION OF THE CONTACT PROBLEM

Here we consider a solid occupying an open bounded domain Ω of a sufficiently regular boundary $\Gamma = \partial\Omega$.

The solid is supposed to have:

- A density on the volume, of force P in Ω
- Homogenous boundary conditions on Γ
- Unilateral contact with a rigid obstacle of equation $x_3 = 0$ on contact surface $\Omega_c = \Omega/\Gamma$.

The displacement is given by:

$$(u(x)).e_3 \geq 0, \text{ in } \Omega_c$$

With (e_1, e_2, e_3) Cartesian base we denote by η the reaction of the obstacle on the solid. The relations leading to a unilateral contact (without friction) are given by:

$$\begin{cases} (u(x)).e_3 \geq 0, \text{ in } \Omega_c \\ (\eta).e_3 \geq 0, \text{ in } \Omega_c \\ (u(x)\eta).e_3 = 0, \text{ in } \Omega_c \end{cases}$$

We use the space $H_0^1(\Omega, R^3)$ of functions in H^1 (Ω, R^3) equals to zero on Γ.

Let us introduce the convex subspace K for the authorized displacements, to be defined as:

$$K=\left\{v\in H_0^1(\Omega,R^3)\ ,(v).e_3\geq 0,\ \text{in}\ \Omega_c\right\}$$

We consider the following variationnal formulation Find:

$$(u,\eta)\in H_0^1(\Omega,R^3)\times H^{-1}(\Omega)$$

Such that:

$$(P_e)\ a(u,v)-c(\eta,v)=l(v),\quad \forall v\in H_0^1(\Omega,R^3)$$

With:

$$c(\eta,v)=\int_{\Omega_c}\eta v dx$$

And the reduced problem becomes:
Find $u\in K$ such that:

$$(P_I)\begin{cases}a(u,v-u)\geq\\ \quad l(v-u)\end{cases}$$

Theorem 2.1

For any solution (u, η) of problem (P_e), u is a solution of problem (P_I).

Proof

Let (u,η) be a solution of problem (P_e) and $u\in K$, $\forall v\in K$ and we have:

$$\langle\eta,v\rangle\geq 0\Leftrightarrow-\langle\eta,v\rangle\leq 0$$

Problem (P_e) leads to:

$$\langle\chi-\eta,u\rangle\geq 0,\ \forall\chi\in K$$

We assume that x = 0:

$$-\langle\eta,u\rangle\geq 0\Leftrightarrow\langle\eta,u\rangle\leq 0$$

By replacing v by v-u in line one of problem (P_e), we get:

$$a(u,v-u)-c(\eta,v-u)=l(v-u)$$

Where:

$$-c(\eta,v-u)=-\langle\eta,v-u\rangle=-\langle\eta,v\rangle+\langle\eta,u\rangle\leq 0$$
$$\Rightarrow a(u,v-u)\geq l(v-u),\ \forall v\in K$$

Let u be a solution of problem (P_I) then (u,η) is a solution of (P_e):

$$a(u,v-u)-l(v-u)\geq 0,\ \forall v\in K$$

By using Green's formula, we get:

$$a(u,v-u)-\langle\eta,v-u\rangle-l(v-u)\geq 0$$

We assume that v = i±φ, with $\varphi\in D(\Omega R^3)$, (i.e., φ is of a compact support), then the integral on the contour is zero:

$$a(u,\phi)=l(\phi),\ \forall\phi$$

The integral on a contact area leads to:

$$\langle\eta,v-u\rangle\geq 0,\ \forall v\in K$$

By assuming that:

$$\begin{cases}v=0\\ v=2u\end{cases}\Rightarrow\langle\eta,u\rangle=0$$

And with the property of convexity of K, we get:

$$\langle\chi-\eta,u\rangle=0\langle\chi,u\rangle-\langle\eta,u\rangle=\langle\chi,u\rangle\geq 0$$

Theorem 2.2

For any $P\in H^{-1}(\Omega, R^3)$, the problem (P_e) has a unique solution $(u,\eta)\in H_0^1(\Omega R^3)\times H^{-1}(\Omega)$

Proof

The existence of the solution u of problem is a direct application of Slimane *et al*. (2004).
Let us consider:

$$L(v)=a(u,v)-l(v)$$

Remark

In problem (P_I), we have:

- if v = 0, then:

$$-a(u,u)\geq -l(u)$$

- if v = 2u, then:

$$a(u,u) \geq l(u) \Rightarrow L(u) = 0$$

The Ker of the form (η, v) is characterized by:

$$V = \left\{ v \in H_0^1\left(\Omega, R^3\right),\ u.e_3 = 0,\ \text{in}\,\Omega \right\}$$

Let v∈V, then v and -v are in K from the problem (P_I) and L(u) = 0, we have:

$$\begin{aligned} &a(u,v) - a(u,u) + b(v,\lambda) - l(v) + l(u) \geq 0 \\ &-l(v) + l(u) \geq 0 \Leftrightarrow\ a(u,v) - l(v) - a(u,u) \\ &-l(u) \geq 0 \Leftrightarrow\ a(u,v) - l(v) \geq 0 \Rightarrow L(u) = 0 \end{aligned}$$

We remplace v by -v in L(u) to get:

$$\begin{aligned} &a(u,v) - l(-v) \geq 0 \Leftrightarrow\ -a(u,v) + l(v) \geq 0 \\ &\Leftrightarrow\ a(u,v) - l(v) \leq 0 \Leftrightarrow\ \ L(u) = 0 \end{aligned}$$

L is of a compact support in V and from the following inf-sup condition:

$$\sup \frac{\langle \eta, v \rangle}{\|v\|} \geq \beta \|\eta\|_{H^{-1}}$$

We can prove that there exists $\eta \in H^{-1}(\Omega)$.

Then (u, η) satisfies line one of problem (P_e).

The definition of K and L(u) = 0, leads to:

$$\langle \chi - \eta, u \rangle = \langle \chi, u \rangle - \langle \eta, u \rangle\ = \langle \chi, u \rangle \geq 0,\ \ \forall \chi \in K$$

This proves the existence of the solution.

Let U_1 and U_2 be two solutions of problem (P_I). With $U_1 = u_1$ and $U_2 = u_2$ then:

$$\begin{aligned} &a(U_1, W - U_1) \geq l(W - U_1),\ \forall W \in K \\ &a(U_2, W - U_2) \geq l(W - U_2), \forall W \in K \end{aligned}$$

By adding that W = U_2 and W = U_1we have:

$$\begin{aligned} &a(U_1, U_2 - U_1) \geq l(U_2 - U_1) a(U_2, U_1 - U_2) + b(U_1 - U_2, \lambda_2) \\ &\geq l(U_1 - U_2) \Leftrightarrow a(U_1, U_1 - U_2) \geq l(U_1 - U_2) \\ &a(U_2, U_1 - U_2) \geq l(U_1 - U_2) \end{aligned}$$

$$\begin{aligned} &a(U_1 - U_2, U_1 - U_2) \leq 0 \\ &\Leftrightarrow a(U_1 - U_2, U_1 - U_2) \leq 0 \\ &\Leftrightarrow \alpha \|U_1 - U_2\|^2 \leq 0\ \Leftrightarrow\ U_1 = U_2 \end{aligned}$$

By the inf-sup condition of problem (P_e) gives us:

$$\forall v \in H_0^1\left(\Omega, R^3\right)\ , \langle \eta_1, v \rangle = \langle \eta_2, v \rangle \Leftrightarrow \eta_1 = \eta_2$$

3. THE DISCRETE PROBLEM

We introduce a discrete subspace V_h of V such that:

$$V_h = \left\{ v_h \in C\left(\Omega, R^3\right),\ v_h \in P_1(k)\ \ v_h = 0, \text{on } \partial\Omega \right\}$$

And dim $V_h < \infty$, therefore there exists a basis: $\{\omega_i\}$, I = 1 to N_h, we can then write:

$$v_h = \sum_{i=1}^{N_h} \beta_i \omega_i$$

Now, let us construct a closed convex subset K_h of V_h such that K_h should be reduced to a finite number of constraints on the β_i:

$$K_h = \left\{ \begin{matrix} v_h \in V_h,\ v.e_3 \geq \varphi \\ \text{at every vertex of each triangle K} \end{matrix} \right\}$$

Then $K_h \subset K$ and $K_h \subset V_h$.

We remark that problem (P_I) is equivalent to find $u_h \in K_h$ such that:

$$(P_h)\ a(u_h, v_h - u_h) \geq l(v_h - u_h),\ \forall v_h \in K_h$$

We assume U = u and W = v.

Theorem 3.1

Let U and U_k be the solutions of problems (P_I) and (P_h), respectively. Let us denote by A∈L (V, V') the map defined, by a(U, W) = (AU, W), then:

$$\|U - U_h\|_V = \begin{bmatrix} \frac{M^2}{\alpha^2} \|U - W_h\|_V^2 + \frac{1}{\alpha} \\ \|P - AU\|_{V'} \\ \|U - W_h\|_V + \|U_h - W\|_V \end{bmatrix}^{\frac{1}{2}}$$

With P is the resultant of the volume force.

Proof

By the definitions of U and W, we have:

$$a(U, U-W) \leq (P, U-W),$$
$$\forall W \in K a(U_h, U_h - W_h) \leq (P, U_h - W_h), \forall W_h \in K_h$$

By adding these inequalities and transposing terms, we obtain:

$$a(U,U) + a(U_h, U_h) \leq (P, U-W)$$
$$+(P, U_h - W_h) + a(U,W) + a(U_h, W_h)$$

By subtracting a(U, U_h)+ a(U_h, U) from both sides and grouping terms and by using the continuity and the coercively of the bilinear form a(U, W), we deduce:

$$\alpha \|U - U_h\|_V{}^2 \leq \begin{bmatrix} \|P - AU\|_{V'} \|U - W_h\|_V \\ +\|P - AU\|_{V'} \|U_h - W\|_V \\ +M\|U - U_h\|_V \|U - W_h\|_V \end{bmatrix}$$

Since:

$$M\|U - U_h\|\|U - W_h\| \leq \frac{M}{\alpha}\|U - W_h\|^2$$

We obtain:

$$\|U - U_h\|_V = \begin{bmatrix} \frac{M^2}{\alpha^2}\|U - W_h\|_V^2 + \frac{1}{\alpha}\|P - AU\|_{V'} \\ \|U - W_h\|_V + \|U_h - W\|_V \end{bmatrix}^{\frac{1}{2}}$$

$$\forall W \in K \text{ and } \forall W_h \in K_h$$

4. NUMERICAL RESULTS

Consider elastic plate with the undeformed rectangle shape (0, 10)×(0, 2). The body force is the gravity force f and the boundary force g is zero on lower and upper side. On the two vertical sides of the beam are fixed (**Fig. 1-3**).

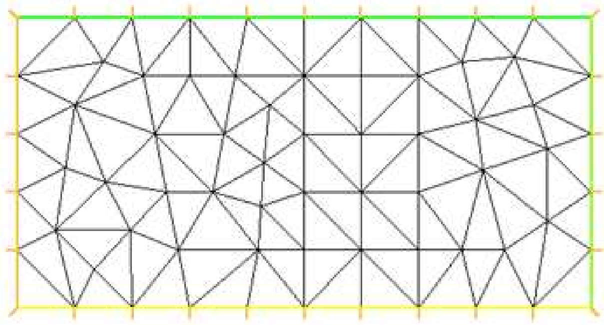

Fig. 1. Mesh

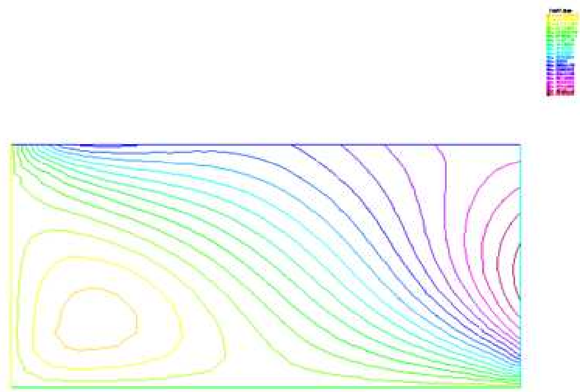

Fig. 2. Isovalue of deplacement u_x

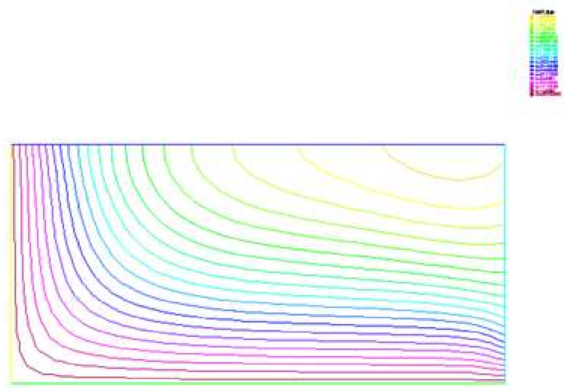

Fig. 3. Isovalue of deplacement u_y

5. CONCLUSION

By starting with the classical model for a deformed elastic solid with a unilateral contact of a rigid body, we proved the existence of solutions for a class of variational inequalities.

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