Journal of Mathematics and Statistics 8 (1): 32-36, 2012
ISSN 1549-3644

An Embedded Explicit Hybrid Method for Ordinary Differential Equations

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Abstract: Problem statement: Many differential systems that appear in practice were special second-order ordinary differential equations of the form y" = f (x, y). In the past research, there was a continuous need for methods for numerically solving these equations. **Approach:** This study describes the derivation and implementation of a pair of embedded explicit hybrid methods for solving non-stiff second-order ordinary differential equations y" = f (x, y). **Results:** It was shown that our method was more efficient than the well-known embedded pair of explicit runge-kutta-nystrom methods for solving some second-order problems. **Conclusion:** Our method can be considered as an alternative for the numerical solution of y" = f (x, y).

Key words: Ordinary Differential Equations (ODEs), Runge Kutta-Nystrom (RKN), multistep methods, numerically solving, hybrid method

INTRODUCTION

There has been great interest in the research of new methods for numerically solving the second-order Ordinary Differential Equations (ODEs) Eq. 1:

$$\mathbf{y}'' = \mathbf{f}(x, \mathbf{y}), \mathbf{y}(x_0) = \mathbf{y}_0, \mathbf{y}'(x_0) = \mathbf{y}'_0 \quad (1)$$

In which the first derivative does not appear explicitly and x_0 denotes the initial point of integration interval. Such problems often arise in engineering and applied sciences such as celestial mechanics, quantum mechanics, elastodynamics, theoretical physics, chemistry and electronics. Problems of the form (1) can be reduced to first-order systems of twice the dimension and solved by using Runge Kutta methods (see for example Babatola *et al*., 2008) or second derivative multistep methods (Parand and Hojjati, 2008). However, this approach is cumbersome and increases computational time (Kayode and Awoyemi, 2010). Thus, it is more efficient to solve the problems directly using Runge Kutta-Nystrom (RKN) methods, multistep methods or block methods (Ken *et al*., 2008). Several authors have proposed hybrid methods which are obtained from the idea underlying both the Runge Kutta and linear multistep methods.

The class of explicit hybrid methods derived by Franco (2006) ranging from fourth to sixth order are applied using a fixed step-size. Such an approach can result in gross inefficiency since the small step-size which must normally be used in the initial part of integration must then be used throughout the integration. In this study, based on explicit hybrid methods in Franco (2006), our attempt is to derive an embedded explicit hybrid method at the cost of five function evaluations at each step and with built-in error estimation suitable for the numerical integration of non-stiff second-order problems of the form (1).

Embedded pairs of hybrid methods: For the numerical solution of (1), we consider the explicit hybrid method:

$$Y_1 = y_{n-1}, Y_2 = y_n$$

$$Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} f(x_n + c_j h, Y_j), i = 3, \ldots, s$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \left[b_1 f_{n+1} b_2 f_n + \sum_{i=3}^{s} b_i f(x_n + c_i h, Y_i) \right]$$

where, f_{n-1} and f_n represent $f(x_{n-1},y_{n-1})$ and $f(x_n,y_n)$ respectively. The method requires s-1 function evaluations or stages at each step of integration. The explicit hybrid method can be represented by the Butcher tableau:

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$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccccc} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ c_3 & a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array}$$

An embedded p(q) pair of hybrid methods is based on the hybrid method (c, A, b) of order p and another hybrid method $(c, A, \bar{b})$ of order q < p, represented by the following tableau:

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & \bar{b}^T \end{array}$$

Embedded pairs of explicit hybrid methods are used in variable step-size algorithm because they provide cheap error estimation. A local error estimation is given by the formula:

$$LTE = \left\| \mathbf{y}_{n+1} - \bar{\mathbf{y}}_{n+1} \right\|$$

where, $\mathbf{y}_{n+1}$ and $\bar{\mathbf{y}}_{n+1}$ are solutions obtained using the higher order and the lower order formula respectively. The LTE is used to control the step-size of which the procedure is given by:

- If tol/div < LTE < div.tol then $h_{n+1} = h_n$
- If LTE ≤ tol/div then $h_{n+1} = 2h_n$
- If LTE ≥ div tol then $h_{n+1} = \frac{1}{2} h_n$ and repeat the step

where, from numerical experiments, div is chosen to be 2^{17}. We do not allow step-size change after each step because it would contribute to unnecessary rounding errors. If the step is acceptable (i.e., tol/div < LTE < div.tol and LTE ≤ tol/div) then we adopt the widely used of performing local extrapolation: although the LTE is the error estimation for the lower order formula, the solutions obtained by using the higher order formula are actually accepted at each point.

MATERIALS AND METHODS

Derivation of the method: The higher order formula is based on our five-stage seventh order explicit hybrid method EHM7(8,7) which has been derived. In order to obtain the four-stage fifth order method, firstly we list out the equations of conditions given in Coleman (2003) as follows Eq. 2-14:

$$b_1 + b_2 + b_3 + b_4 + b_5 = 1 \quad (2)$$

$$-b_1 + b_3 c_3 + b_4 c_4 + b_5 c_5 = 0 \quad (3)$$

$$b_1 + b_3 c_3^2 + b_4 c_4^2 + b_5 c_5^2 = \frac{1}{6} \quad (4)$$

$$\begin{aligned} & b_3 a_{31} + b_3 a_{32} + b_4 a_{41} + b_4 a_{42} + b_4 a_{42} + b_4 a_{43} + b_5 a_{51} \\ & + b_5 a_{52} + b_5 a_{53} + b_5 a_{54} = \frac{1}{12} \end{aligned} \quad (5)$$

$$\begin{aligned} & -b_1 + b_3 c_3^3 + b_4 c_4^3 + b_5 c_5^3 = 0 \\ & b_3 c_3 a_{31} + b_3 c_3 a_{32} + b_4 c_4 a_{41} + b_4 c_4 a_{42} + b_4 x = c_4 a_{43} \end{aligned} \quad (6)$$

$$+ b_5 c_5 a_{51} + b_5 c_5 a_{52} + b_5 c_5 a_{53} + b_5 c_5 a_{54} = \frac{1}{12} \quad (7)$$

$$\begin{aligned} & -b_3 a_{31} - b_4 a_{41} + b_4 a_{43} c_3 - b_5 a_{51} \\ & + b_5 a_{53} c_3 + b_5 a_{54} c_4 = 0 \end{aligned} \quad (8)$$

$$b_1 + b_3 C_3^4 + b_4 c_4^4 + b_5 c_5^4 = \frac{1}{15} \quad (9)$$

$$\begin{aligned} & b_3 c_3^2 a_{31} + b_3 c_3^2 a_{32} + b_4 c_4^2 a_{41} + b_4 c_4^2 a_{42} + \\ & b_4 c_4^2 a_{43} + b_5 c_5^2 a_{51} + b_5 c_5^2 a_{52} + b_5 c_5^2 a_{53} \\ & + b_5 c_5^2 a_{54} = \frac{1}{30} \end{aligned} \quad (10)$$

$$\begin{aligned} & -b_3 c_3 a_{31} - b_4 c_4 a_{41} + b_4 c_4 a_{43} c_3 - b_5 c_5 a_{51} \\ & + b_5 c_5 a_{53} c_3 + b_5 c_5 a_{54} c_4 = -\frac{1}{60} \end{aligned} \quad (11)$$

$$\begin{aligned} & b_3 a_{31}^2 + 2 b_3 a_{31} a_{32} + b_3 a_{32}^2 + b_4 a_{41}^2 + 2 b_4 a_{41} a_{42} \\ & + 2 b_4 a_{41} a_{43} + b_4 a_{42}^2 + 2 b_4 a_{42} a_{43} + b_4 a_{43}^2 \\ & + b_5 a_{51}^2 + 2 b_5 a_{51} a_{52} + 2 b_5 a_{51} a_{53} + 2 b_5 a_{54} \\ & + 2 b_5 a_{53} a_{54} + b_5 a_{54}^2 = \frac{7}{120} \end{aligned} \quad (12)$$

$$\begin{aligned} & b_3 a_{31} + b_4 a_{41} + b_4 a_{43} c_3^2 + b_5 a_{51} \\ & + b_5 a_{53} c_3^2 + b_5 a_{54} c_4^2 = \frac{1}{180} \end{aligned} \quad (13)$$

$$\begin{aligned} & b_4 a_{43} a_{31} + b_4 a_{43} a_{32} + b_5 a_{53} a_{31} + b_5 a_{53} a_{32} \\ & + b_5 a_{54} a_{41} + b_5 a_{54} a_{42} + b_5 a_{54} a_{43} = \frac{1}{360} \end{aligned} \quad (14)$$

Then, b in the Eq. 2-14 is replaced by $\bar{b}$. Next, putting values of the coefficients c_3, c_4, c_5, a_{31}, a_{32}, $a_{41}, a_{42}, a_{43}, a_{51}, a_{52}, a_{53}$ and a_{54} obtained from the derivation of EHM7(8,7) method into the resulting equations and solving for $\bar{b}$ we get:

$$\bar{b}_1 = \frac{36479}{257748} - \frac{1350}{21479}\sqrt{5}$$
$$\bar{b}_2 = \frac{42535}{57042} - \frac{2500}{28521}\sqrt{5}$$
$$\bar{b}_3 = \frac{28609200}{3112602559} + \frac{178697500}{3112602559}\sqrt{5}$$
$$\bar{b}_4 = \frac{33521}{942252} - \frac{1250}{78521}\sqrt{5}$$
$$\bar{b}_5 = \frac{22006250000000000000000}{1497229238426516076601} + \frac{15625000000000000000000}{14722923538426516076601}$$

The fifth order method has an interval of absolute stability (0, 2.643). The phase-lag and the dissipation error for this method are respectively given by:

$$\varphi(H) = \left(-\frac{28451909}{169920000000}\sqrt{5} + \frac{269477989}{713664000000}\right)H^7 + O(H^9)$$

and:

$$d(H) = \left(\frac{1266071}{53100000000}\sqrt{5} - \frac{20789}{18880000}\right)H^6 + O(H^8)$$

Thus, the fifth order method has phase-lag of order six and is dissipative of order five. The new embedded method is denoted by EHM7(5). We cannot get the 7(6) pair of hybrid methods due to the limited number of coefficients to turn into free parameters.

RESULTS AND DISCUSSION

Problems tested: Our method has been applied to the following problems to provide numerical comparisons with the well-known embedded 6(4) pair of Runge Kutta Nystrom methods derived in Dormand *et al*. (1987):

Problem 1:

$$y'' = -100y + 99\sin(x), (y)(0) = 1, y'(0) = 11, x \in [0,10]$$

Theoretical solution:

$$y(x) = \cos(10x) + \sin(10x) + \sin(x)$$

Problem 2:

$$y_1'' = 4x^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, y_1(0) = 1, y_1'(0) = 0$$
$$y_2'' = -4x^2 y_2 + \frac{2y_1}{\sqrt{y_1^2 + y_2^2}}, y_2(0) = 0, y_2'(0) = 0, x \in [0,5]$$

Theoretical solution:

$$y_1(x) = \cos(x^2), y_2(x) = \sin(x^2)$$

Problem 3:

$$\frac{d^2 y_1(x)}{dx^2} = -13y_1(x) + 12y_2(x) + f_1(x)$$
$$y_1(0) = 1, y_1'(0) = -4$$
$$\frac{d^2 y_2(x)}{dx^2} = 12y_1(x) - 13y_2(x) + f_2(x)$$
$$y_2(0) = 0, y_2'(0) = 8, x \in [0,200]$$

with:

$$f_1(x) = 9\cos(2x) - 12\sin(2x)$$
$$f_2(x) = -12\cos(2x) + 9\sin(2x)$$

Theoretical solution:

$$y_1(x) = \sin(x) - \sin(5x) + \cos(2x)$$
$$y_2(x) = \sin(x) + \sin(5x) + \sin(2x)$$

Problem 4:

$$\frac{d^2 y_1(x)}{dx^2} + \lambda^2 y_1(x) = f''(x) + \lambda^2 f(x),$$
$$y_1(0) = a + f(0), y_1'(0) = f'(0)$$
$$\frac{d^2 y_2(x)}{dx^2} + \lambda^2 y_2(x) = f''(x) + \lambda^2 f(x)$$
$$y_2(0) = f(0), y_2'(0) = \lambda a + f'(0), x \in [0, 3\phi]$$

Theoretical solution:

$$y_1(x) = a\mathrm{co}(s\lambda x) + f(x), y_2(x) = a\sin(\lambda x) + f(x)$$

In this research, f(x) is chosen to be $e^{-0.05x}$ and a and λ be 0.1 and 20 respectively.

Problem 5:

$$y'' = -y - y^3 + B\cos(vx), y(0) = 0.200426728067$$
$$y'(0) = 0, x \in [0,20]$$

where, $B = \frac{1}{500}$ and v = 1.01. The exact solution computed by the Galerkin method with a precision 10^{-12} of the coefficients is given by:

$$y(x) = A_1 \cos(vx) + A_3 \cos(3vx) + A_5 \cos(5vx) + A_7 \cos(7vx) + A_9 \cos(9vt)$$

where:

$$A_1 = 0.200179477536, A_3 = 2.46946143.10^{-4}$$
$$A_5 = 3.04014.10^{-7}, A_7 = 3.74.10^{-10}$$
$$A_9 = 0.000000000000.$$

The following are codes that have been used in the comparisons:

- EHM7(5): Embedded explicit hybrid method 7(5) pair derived in this study
- RKN6(4)D: Embedded explicit RKN method 6(4) pair with six stages per step derived by Dormand *et al*. (1987)

Since RKN6(4)D has FSAL (first same as last) property, therefore it can be regarded as possessing five stages per step. The step-size for RKN6(4)D code is controlled by using the similar procedure as for EHM7(5) code.

Graphs of natural logarithm of maximum global error (log (MAXGE)) versus TIME of each code are displayed in Fig. 1-5.

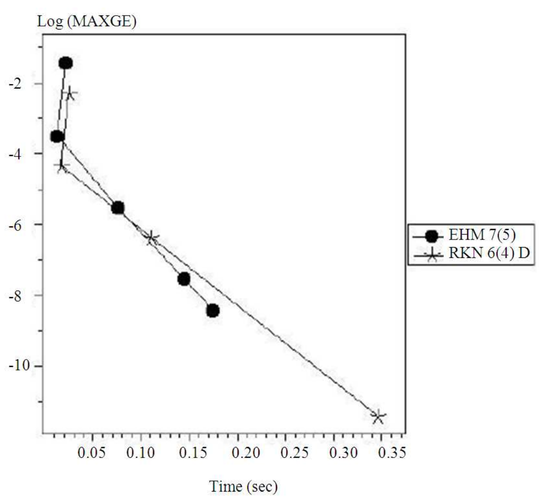

Fig. 1: Log (MAXGE) versus TIME graphs of EHM7 (5) and RKN6(4)D for Problem 1

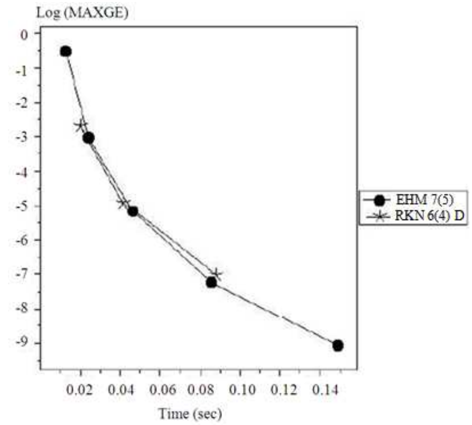

Fig. 2: Log (MAXGE) versus TIME graphs of EHM7 (5) and RKN6(4)D for Problem 2

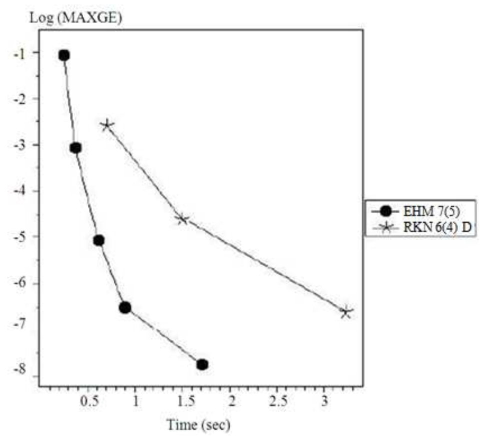

Fig. 3: Log (MAXGE) versus TIME graphs of EHM7 (5) and RKN6(4)D for Problem 3

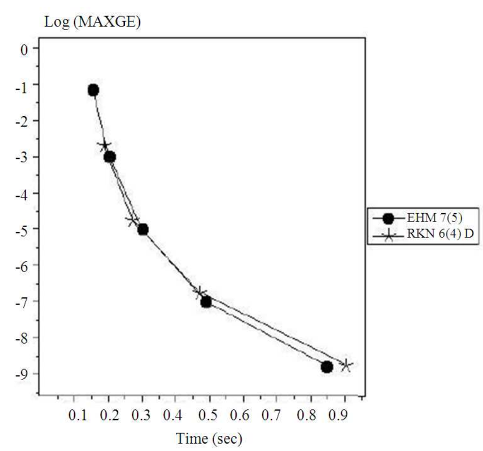

Fig. 4: Log (MAXGE) versus TIME graphs of EHM7 (5) and RKN6(4)D for Problem 4

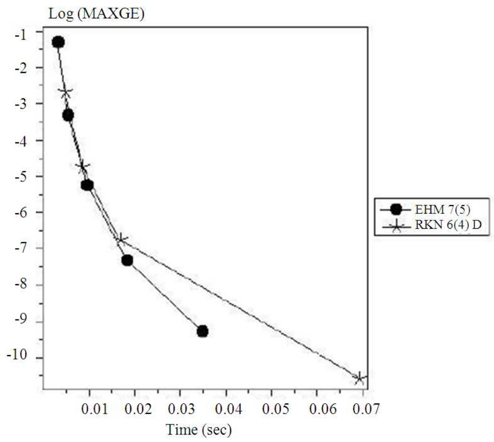

Fig. 5: Log (MAXGE) versus TIME graphs of EHM7 (5) and RKN6(4)D for Problem 5

CONCLUSION

For Problems 3 and 5, it is clear that EHM7(5) is more efficient than RKN6(4)D method while for the rest of the problems, EHM7(5) and RKN6(4)D methods are both competitive. In conclusion, the new embedded explicit hybrid method 7(5) pair is capable for solving any physical problems involving system of second-order ordinary equations of the form **y**”= **f**(x,**y**). All codes are designed using Microsoft Visual C++ version 6.0 in HP computer with Intel (R) Core (TM) 2 Duo CPU P8600@2.40GHz.

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