

Distribution of Records Defined on Ordered Words Representing Lattice Paths

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Abstract: Problem statement: Let the sequence i_1, i_2, \dots, i_n , denoted by S_n^n be an increasing ordered word of length n taken from the set of the n positive integers $S = \{1, 2, \dots, n\}$, $m, n \in \mathbb{N}^+$, $m \geq n$. **Approach:** That is $1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq n$. Treating S_n^n as a sequence of weak records $\{L_j = i_j\}$, $i, j = 1, 2, \dots, n$, the distribution of the single weak record as well as the joint distribution of weak records were found before. **Results:** By defining the notion of strong records on the sequence $\{L_j = i_j\}$, the distribution of a single strong record was found for $m=n$. In another aspect, it can be shown that the lattice path in the plane from $(0,1)$ to (m, n) , consisting of unit segments up and to the right, can be represented by a sequence i_1, i_2, \dots, i_m where $1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n$, $m, n \in \mathbb{N}^+$, $m \geq n$. That is, such lattice paths can be represented, in one to one correspondence, by ordered increasing words of length m taken from the set S . **Conclusion/Recommendations:** In this article, we are going to extend the notion of weak and strong records to these sequences representing lattice paths for $m > n$ and obtain their distributions. This result allows us to study lattice paths via ordered words of non negative integers.

Key words: Lattice paths, increasing ordered words, random variables, probability distribution

INTRODUCTION

An ordered word of length m taken from the set $S = \{1, 2, \dots, n\}$ Alahmadi (2009), is a set of m integers i_1, i_2, \dots, i_m such that $1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n$, $m, n \in \mathbb{N}^+$. For $n=m$, the element i_j , $i, j=1, 2, \dots, n$. is denoted by the weak record L_i the probability distribution of a single L_j as well as the joint probability distribution of L_1, L_2, \dots, L_k , where $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$, were given by the following theorem, for other results on ordered sequences El-Faheem and Mahmoud (2010), Khidr and Radwan (2000).

Theorem 1: Let $P(L_i = j)$ be the probability that the integer j falls in the i^{th} place in the sequence $j_1 j_2 \dots j_n$, $i = 1, 2, \dots, n$. then we have Eqn. 1:

$$P(L_i = j) = \binom{i+j-2}{j-1} \binom{2n-i-j}{n-j}, j=1, 2, \dots, n \quad (1)$$

Similarly, if $P(L_{i_1} = j_1, L_{i_2} = j_2, \dots, L_{i_k} = j_k)$ be the the probability that the numbers j_1, j_2, \dots, j_k falling in the locations i_1, i_2, \dots, i_k , respectively, then Eq. 2:

$$P(L_{i_1} = j_1, L_{i_2} = j_2, \dots, L_{i_k} = j_k) = \binom{i_1+j_1-2}{j_1-1} \binom{i_2+j_2-i_1-j_1-1}{j_2-j_1-1} \dots \binom{i_k+j_k-i_{k-1}-j_{k-1}-1}{j_k-j_{k-1}-1} \binom{2n-i_k-j_k}{n-j_k} \quad (2)$$

In the case $m > n$, similar expressions are given for $P(L_{i_1} = j_1, L_{i_2} = j_2, \dots, L_{i_k} = j_k) \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n, \leq k \leq m$.

We notice that $1 \leq L_{i_1} \leq L_{i_2} \leq \dots \leq L_{i_k} \leq n$, thus they form some kind of weak records, the strong record on the sequence $j_1 j_2 \dots j_n$ or alternately on $L_{i_1}, L_{i_2}, \dots, L_{i_k}$ was introduced as a parallel to record values in the continuous case. For records from continuous distributions, Ahsanullah and Ragab (2006) Ahsanullah (1995) for discrete records, Dembinska (2007).

MATERIALS AND METHODS

Definition 1: For $m \geq n$.

Let $R(1)=1$ and $R(i+1) = \min\{k > R(i) : L_k > X_{R(i)}\}$ when L_k exists, otherwise $R(i+t) = R(i)$, $t=1, 2, \dots, m-i$, then $X_{R(1)}, X_{R(2)}, \dots, X_{R(m)}$ are called the strong record values on L_1, L_2, \dots, L_m . and $P(X_{R(1)} = k)$ is given by the following theorem Eqn. 3.

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Theorem 2:

$$P(X_{R(i)} = k) = \binom{k-1}{i-1} \binom{2n-k-1}{n-i} + \sum_{r=0}^{i-2} \binom{n-1}{r} \binom{k-1}{r} \quad i=1,2,\dots,n; k=1,2,\dots,n \tag{3}$$

In this article, we generalize Theorem 2 for $m > n$. to get.

Theorem 3: For $m > n$, the distribution of the record value $X_{R(i)} = k$ is equal to Eqn. 4:

$$P(X_{R(i)} = k) = \binom{k-1}{i-1} \binom{m+n-k-1}{m-i} + \sum_{r=0}^{i-2} \binom{m-1}{r} \binom{k-1}{r} \quad i=1,2,\dots,m; k=1,2,\dots,n \tag{4}$$

The case $m < n$, can be obtained from (3) by interchanging m and n in (2).

Proof of theorem 3: We call the sequence of random variables $L_1 L_2 \dots L_m$ (alternatively the ordered word $j_1 j_2 \dots j_m$) the original sequence and its corresponding sequence of the record values, $X_{R(1)} X_{R(2)} \dots X_{R(m)}$, the reduced sequences. For example, for $n = 6, m = 7$, the sequence of random variables 2234666 is an original sequence and its corresponding records 2346666 is the corresponding reduced sequence. We notice that the value of the i^{st} record $X_{R(i)}$ is equal to the i^{st} entry in the reduced sequence, $I = 1, 2, \dots, m$ Therefore, to obtain the number of original sequences $L_1 L_2 L_3 \dots L_m$, or alternatively $j_1 j_2 \dots j_m$, contributing to the event $X_{R(i)} = k, i > 1$ (for $i=1$, by definition 1, $P(X_{R(1)} = k) = P(L_1 = k)$, we have to look for the reduced sequences which have the i^{st} entry equal to $k, k=1,2,\dots,n$. These reduced sequences are one of two types.

Type 1, sequences which have the first r entries $X_{R(1)}, X_{R(2)} \dots X_{R(r)}$ form an strictly increasing ordered word from the set $\{1, 2, \dots, (k-1)\}, 0 \leq r \leq (i-2), i > 1$. The remaining elements of the sequence are all equal to k . These sequences will be of the form:

$$X_{R(1)} X_{R(2)} \dots X_{R(r)} X_{R(r+1)} X_{R(r+2)} \dots X_{R(m)} = j_1 j_2 \dots j_r j_{(r+1)} j_{(r+2)} \dots j_m$$

With:

$$1 \leq j_1 < j_2 < \dots < j_r \leq (k-1), j_s = k, s = (r+1), (r+2), \dots, m$$

Or what is the same:

$$X_{R(1)} X_{R(2)} \dots X_{R(r)} X_{R(r+1)} X_{R(r+2)} \dots X_{R(m)} = j_1 j_2 \dots j_r k k \dots k \tag{5}$$

When $r = 0$, (5), reduces to the sequence $kkk \dots k$

Type 2, sequences with the first $(i-1)$ entries are formed of strictly increasing words from the set $\{1, 2, \dots, (k-1)\}$, the i^{st} entry equals k and the remaining $(m-i)$ entries are formed of strictly increasing ordered word formed from the set $\{k, (k+1), \dots, n\}$. These sequences are of the form Eqn. 6:

$$X_{R(1)} X_{R(2)} \dots X_{R(i-1)} X_{R(i)} X_{R(i+1)} \dots X_{R(m)} = j_1 j_2 \dots j_{(i-1)} k j_{(i+1)} \dots j_m \text{ with } 1 \leq j_1 < j_2 < \dots < j_{(i-1)} \leq (k-1), j_i = k, k \leq j_{(i+1)} \leq \dots \leq j_m \leq n \tag{6}$$

After determining the reduced sequences that contribute to the event $X_{R(i)} = k, i > 1, k=1,2,\dots,n$. as in (1-2), we have to look for the original sequences that give these reduced sequences when applying definition 3.

For the reduced sequences in (2.1), we can construct the original sequences that give them as follows: From the set of integers $\{1, 2, \dots, (k-1)\}$, we select r integers without replacement in $\binom{k-1}{r}$ ways, denote these r integers $j_1, j_2, \dots, j_r, 0 \leq r \leq (i-1)$. From the m places reserved for the sequence $L_1 L_2 L_3 \dots L_m$, we save the last location for the integer k and choose from the remaining $(n-1)$ places, r places for the previously selected r integers in $\binom{m-1}{r}$ ways. Place these r integers in the r selected places to get the following arrangement below:

$$j_i \quad j_2 \quad j_r \quad k$$

Now we fill in the empty places by the integer succeeding it to get the following arrangement:

$$j_1 j_1 j_1 j_2 j_2 j_2 \dots j_r j_r j_r k k k k K k$$

Clearly, the above arrangement, as original sequence, gives a reduced sequence as in (1). The number of such original sequences, for particular r is $\binom{n-1}{r} \binom{k-1}{r}$, Summing over $r = 0, 2, \dots, (i-2)$, we see that the number of original sequences that give reduced sequences as in (7) is equal to:

$$\sum_{r=0}^{i-2} \binom{m-1}{r} \binom{k-1}{r} \quad (7)$$

For the reduced sequences in (8), we can see that the original sequences that can give the reduced sequences of type 2, as in (8), are of the form as below:

$$j_1 \dots j_1 \ j_2 \dots j_2 \dots j_{(i-1)} \dots j_{(i-1)} \ k \ j_{(i+1)} \dots j_m$$

Where:

$$\begin{aligned} 1 \leq j_1 < j_2 < \dots < j_{(i-1)} \leq (k-1), j_i = k, \\ k \leq j_{(i+1)} \leq \dots \leq j_m \leq n \end{aligned} \quad (8)$$

The number of the arrangements (2.4) can be obtained as follows: chose (i-1) strictly increasing positive integers, $1 \leq j_1 < j_2 < \dots < j_{(i-1)} \leq (k-1)$, out the set $\{1, 2, \dots, k\}$ in $\binom{k-1}{i-1}$ ways. Then we take the first s places, $1 \leq s \leq m$, we save the s^{th} place for the integer k and choose from the remaining (s-1) places, (i-1) places for the (i-1) integers, $1 \leq j_1 < j_2 < \dots < j_{(i-1)} \leq (k-1)$ and fill in the empty (s-i) places, as before, by the integers succeeding each. This gives us the first part of the scheme (9). To obtain the arrangement (9), we fill in the remaining (m-s) places by ordered words of length (m-s) formed from the set $\{k, k+1, \dots, n\}$.

The total number of original sequences of type 2 will be equal to:

$$\binom{k-1}{i-1} \sum_{s=i}^m \binom{s-1}{i-1} \binom{m-s+n-k-1}{m-s} \quad (9)$$

The sum in (10) is equal to:

$$\binom{m+n-k-1}{m-i}$$

thus (5) equals to:

$$\binom{k-1}{i-1} \binom{m+n-k-1}{m-i} \quad (10)$$

From (3) and (6), the theorem follows.

RESULTS

Theorem 2 is an extension to theorem 1 and the following Table 1 can be used to justify the theorem.

Table 1: $\binom{m+n-1}{m} P(X_{R(i)} = k)$

k/I	1	2	3	4	5
1	210	84	28	7	1
2	1	127	113	64	25
3	1	7	83	124	115
4	1	7	28	99	195
5	1	7	28	84	210
6	1	7	28	84	210
7	1	7	28	84	210

DISCUSSION

It is interesting to proof a similar theorem to theorem 1 and 2, in case the sequence $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$ represents the order statistics from the discrete uniform distribution on $S = \{1, 2, \dots, n\}$. The result if obtained, can be used together with current result in testing hypotheses concerning records.

CONCLUSION

The study of lattice paths is closely related to the study of increasing ordered words of non negative integers, this relation can facilitate the study of either via the other.

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