Journal of Mathematics and Statistics 7 (2): 107-111, 2011
ISSN 1549-3644


# Fuzzy Subalgebras and Fuzzy T-ieals in TM-Algebras

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**Abstract:** In this study, we introduce the concepts of fuzzy subalgebras and fuzzy ideals in TM-algebras and investigate some of its properties. **Problem statement:** Let X be a TM-algebra, S be a subalgebra of X and I be a T-ideal of X. Let μ and v be fuzzy sets in a TM-algebra X. **Approach:** Define the upper level subset $\mu_t$ of μ and the cartesian product of μ and v from X×X to [0,1] by minimum of μ (x) and v (y) for all elements (x, y) in X×X. **Result:** We proved any subalgebra of a TM-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X and $\mu_t$ is a T-ideal of X. Also we proved, the cartesian product of μ and v is a fuzzy T-ideal of X×X. **Conclusion:** In this article, we have fuzzified the subalgebra and ideal of TM-algebras into fuzzy subalgrbra and fuzzy ideal of TM-algebras. It has been observed that the TM-algebra satisfy the various conditions stated in the BCC/ BCK algebras. These concepts can further be generalized.

**Key words:** TM-algebra, fuzzy subalgebra, fuzzy ideals, homomorphism, cartesian product, level subset, conditions stated

## INTRODUCTION

Isaki and Tanaka introduced two classes of abstract algebras BCI-algebras and BCK-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI- algebra. Hu and Li introduced a wide class of abstract algebra namely BCH- algebras. Zadeh (1965), introduced the notion of fuzzy sets in 1965. This concept has been applied to many mathematical branches. Xi applied this concept to BCK-algebra. Dudek and Jun (2001) fuzzified the ideals in BCC-algebras. Jun (2009) contributed a lot to develop the theory of fuzzy sets.

We (Megalai and Tamilarasi, 2010) introduced a new notion called TM-algebra, which is a generalization of Q/BCK / BCI /BCH-algebras and investigated some properties. In this study, we introduce the concepts of fuzzy subalgebras and fuzzy T-ideals in TM-algebra and investigate some of their properties.

## MATERIALS AND METHODS

Certain fundamental definitions that will be used in the sequel are described.

**Preliminaries:**

**Definition 1:** A BCK-algebra is an algebra (X,*, 0) of type (2, 0) satisfying the following conditions:

- $(x*y)*(x*z) \le z*y$
- $x*(x*y) \le y$
- $x \le x$,
- $x \le y$ and $y \le x$ imply $x = y$,
- $0 \le x$ implies $x = 0$, where $x \le y$ is defined by
- $x*y = 0$ for all $x, y, z \in X$.

**Definition 2:** Let I be a non- empty subset of a BCK-algebra X. Then I is called a BCK-ideal of X if:

- $0 \in I$,
- $x*y \in I$ and $y \in I$ imply $x \in I$, for all $x, y \in x$

**Definition 3:** A TM-algebra (X,*,0) is a non-empty set X with a constant "0" and a binary operation "* " satisfying the following axioms:

- $x*0 = x$
- $(x*y)*(x*z) = z*y$, for any $x, y, z \in X$

In X we can define a binary relation $\le$ by $x \le y$ if and only if $x*y = 0$.

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**Definition 4:** Let S be a non-empty subset of a TM-algebra X. Then S is called a subalgebra of X if $x * y \in S$, for all $x, y \in X$.

**Definition 5:** Let (X,*, 0) be a TM-algebra. A non-empty subset I of X is called an ideal of X if it satisfies

- $0 \in I$
- $x * y \in I$ and $y \in I$ imply $x \in I$, for all $x, y \in X$.

**Definition 6:** An ideal A of a TM-algebra X is said to be closed if $0 * x \in A$ for all $x \in A$.

**Definition 7:** Let (X,*,0) be a TM-algebra. A non-empty sub set I of X is called a T- ideal of X if it satisfies

- $0 \in I$
- $(x*y)*z \in I$ and $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.

**Fuzzy subalgebras:**

**Definition 8:** Let X be a non-empty set. A mapping $\mu : x \to [0,1]$ is called a fuzzy set in X. The complement of μ, denoted by $\bar{\mu}(x) = 1 - \mu(x)$, for all $x \in X$.

**Definition 9:** A fuzzy set μ in a TM-algebra X is called a fuzzy subalgebra of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}, \text{for all } x, y \in X.$$

**Definition 10:** Let μ be a fuzzy set of a set X. For a fixed $t \in [0,1]$, the set $\mu_t = \{ x \in X / \mu(x) \geq t\}$ is called an upper level of μ.

**Fuzzy T-ideals in TM-algebras:**

**Definition 11:** A fuzzy subset μ in a TM-algebra X is called a fuzzy ideal of X, if:

- (i) $\mu(0) \geq \mu(x)$
- (ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y, z \in X$

**Definition 12:** A fuzzy subset μ in a TM-algebra X is called a fuzzy T-ideal of X, if:

- $\mu(0) \geq \mu(x)$
- $\mu(x*z) \geq \min\{ \mu((x*y)*z), \mu(y) \}$, for all $x, y, z \in X$

## RESULTS

**Lemma 13:** If μ is a fuzzy subalgebra of a TM-algebra X, then $\mu(0) \geq \mu(x)$ for any $x \in X$.
Proof: Since $x * x = 0$ for any $x \in X$, then:

$$\mu(0) = \mu(x*x) \geq \min\{ \mu(x), \mu(x) \} = \mu(x).$$

This completes the proof.

**Theorem 14:** A fuzzy set μ of a TM-algebra X is a fuzzy subalgebra if and only if for every $t \in [0,1]$, $\mu_t$ is either empty or a subalgebra of X.

**Proof:** Assume that μ is a fuzzy subalgebra of X and $\mu_t \neq \varphi$. Then for any $x,y \in \mu_t$, we have:

$$\mu(x*y) \geq \min\{ \mu(x), \mu(y) \} \geq t.$$

Therefore $x*y \in \mu_t$.
Hence $\mu_t$ is a subalgebra of X.
Conversely, $\mu_t$ is a subalgebra of X.
Let $x, y \in X$. Take $t = \min\{ \mu(x), \mu(y) \}$.

Then by assumption $\mu_t$ is a subalgebra of X implies:

$$x*y \in \mu_t$$

Therefore $\mu(x*y) \geq t = \min\{ \mu(x), \mu(y) \}$.
Hence μ is a subalgebra of X.

**Theorem 15:** Any subalgebra of a TM-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.

**Proof:** Let μ be a subalgebra of a given TM-algebra X and let μ be a fuzzy set in X defined by:

$$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

where, $t \in (0,1)$ is fixed. It is clear that $\mu_t = A$.

Now we will prove that such defined μ is a fuzzy subalgebra of X.

Let $x, y \in X$. If $x, y \in A$ then also $x * y \in A$.

Hence $\mu(x) = \mu(y) = \mu(x*y) = t$ and

$$\mu(x*y) \geq \min\{ \mu(x), \mu(y) \}.$$

If $x, y \notin A$ then $\mu$ (x) = $\mu$ (y) = 0 and in the consequence $\mu$ (x*y) $\geq$ min{ $\mu$ (x) , $\mu$ (y) } = 0.

If at most one of x, y belongs to A, then at least one of $\mu$ (x) and $\mu$ (y) is equal to 0.

Therefore, min { $\mu$ (x) , $\mu$ (y) } = 0 so that:

$\mu$ (x*y) $\geq$ 0, which completes the proof

**Theorem 16:** Two level subalgebras $\mu_s$, $\mu_t$ (s < t ) of a fuzzy subalgebra are equal if and only if there is no $x \in X$ such that s $\leq$ $\mu$ (x) < t.

**Proof:** Let $\mu_s = \mu_t$ for some s < t. If there exits $x \in X$ such that s $\leq$ $\mu$ (x) < t, then $\mu_t$ is a proper subset of $\mu_s$, which is a contradiction.

Conversely, assume that there is no $x \in X$ such that s $\leq$ $\mu$ (x) < t. If $x \in \mu_s$, then $\mu$ (x) $\geq$ s and $\mu$ (x) $\geq$ t, since $\mu$ (x) does not lie between s and t. Thus $x \in \mu_t$, which gives $\mu_s \subseteq \mu_{t-}$ Also $\mu_t \subseteq \mu_s$. Therefore $\mu_s = \mu_t$

**Theorem 17:** Every fuzzy T-ideal $\mu$ of a TM-algebra X is order reversing, that is if $x \leq y$ then:

$\mu$ (x) $\geq$ $\mu$ (y) for all x, y $\in$ X.

**Proof:** Let $x.y \in X$ such that $x \leq y$.

Therefore x*y = 0.

Now, $\mu$ (x) = $\mu$ (x*0)

$\geq$ min{ $\mu$ ( ( x*y )* 0), $\mu$ (y) }

= min{ $\mu$ (0*0) , $\mu$ (y) }

= min{ $\mu$ (0) , $\mu$ (y) }

= $\mu$ (y).

**Theorem 18:** A fuzzy set $\mu$ in a TM-algebra X is a fuzzy T-ideal if and only if it is a fuzzy ideal of X.

**Proof:** Let $\mu$ be a fuzzy T-ideal of X
Then (i) $\mu$ (0) $\geq$ $\mu$ (x) and
(ii) $\mu$ ( x*z) $\geq$ min { $\mu$ ( (x*y) *z) , $\mu$ (y) } for all $x, y, z \in X$.
By putting z = 0 in (ii) we have
$\mu$ ( x) $\geq$ min { $\mu$ (x*y) , $\mu$ (y) .
Hence $\mu$ is a fuzzy ideal of X.
Conversely, $\mu$ is a fuzzy ideal of X.

Then:

$\mu$ ( x*z) $\geq$ min { $\mu$ ( (x*z)* y) , $\mu$ (y) }

= min { $\mu$ ( (x*y) *z) , $\mu$ (y) }, which proves the result.

**Theorem 19:** Let $\mu$ be a fuzzy set in a BCK-algebra X. Then $\mu$ is a fuzzy T-ideal if and only if $\mu$ is a fuzzy BCK-ideal.

**Proof:** Since every BCK-algebra is a TM-algebra, every fuzzy T-ideal is a fuzzy ideal of a TM-algebra and hence a fuzzy BCK-ideal.
Conversely, assume that $\mu$ be a BCK-ideal of X.

Then:

$\mu$ ( x*z) $\geq$ min { $\mu$ ( (x*z)* y) , $\mu$ (y) }

= min { $\mu$ ( (x*y)* z) , $\mu$ (y)}.

Hence $\mu$ is a fuzzy T-ideal of X.

**Theorem 20:** Let $\mu$ be a fuzzy set in a TM-algebra X and let $t \in \mathrm{Im}(\mu)$. Then $\mu$ is a fuzzy T-ideal of X if and only if the level subset:

$\mu_t$ = { x $\in$ X / $\mu$ (x) $\geq$ t }

is a T-ideal of X, which is called a level T-ideal of $\mu$.

**Proof:** Assume that $\mu$ is a fuzzy T-ideal of X.
Clearly $0 \in \mu_t$

Let (x*y) *z $\in$ $\mu_t$ and $y \in \mu_{t-}$

Then $\mu$ ((x*y) *z) $\geq$ t and $\mu$ (y) $\geq$ t.

Now $\mu$ (x*z) $\geq$ min { $\mu$ ((x*y)* z) , $\mu$ (y)}

$\geq$ {t, t} = t

Hence $\mu_t$ is T-ideal of X.

Conversely, let $\mu_t$ is T-ideal of X for any $t \in [0,1]$.
Suppose assume that there exist some $x_0 \in X$ such that $\mu$ (0) < $\mu$ ( $x_0$ ):

Take $s = \frac{1}{2}[\mu(0) + \mu(x_0)\ ]$

$\Rightarrow s < \mu(x_0)\text{and}0 \leq \mu(0) < s < 1$

$\Rightarrow x_0 \in \mu_s\text{and}0 \notin \mu_s$ a contradiction, since

$\mu_s$ is a T-ideal of X.

Therefore, $\mu(0) \geq \mu(x)$ for all $x \in X$

If possible, assume that $x_0, y_0, z_0 \in X$ such that $\mu(x_0 * z_0) \geq \min\{\mu((x_0 * y_0) * z_0), \mu(y_0)\}$:

Take $s = \frac{1}{2}[\mu(x_0 * z_0) + \min\{\mu(x_0 * y_0) * z_0), \mu(y_0)\}$

$\Rightarrow s > \mu(x_0 * z_0)$

and:

$s < \min\{\mu((x_0 * y_0) * z_0), \mu(y_0)\}$

$\Rightarrow s > \mu(x_0 * z_0)$, $s < \mu((x_0 * y_0) * z_0)$ and $s < \mu(y_0)$

$\Rightarrow x_0 * z_0 \notin \mu_s$, a contradiction, since $\mu_s$ is a T-ideal of X.

Therefore, $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}$ for any $x, y, z \in X$.

**Cartesian product of fuzzy T-ideals of TM-algebras:**

**Definition 21:** Let μ and v be the fuzzy sets in a set X. The Cartesian product $\mu \times v: X \times X \rightarrow [0,1]$ is defined by:

$(\mu \times v)(x,y) = \min\{\mu(x), v(y)\}$ for all $x, y \in X$

**Theorem 22:** If μ and v are fuzzy T-ideals in a TM-algebra X, then μ ×v is a fuzzy T-ideal in X×X.

**Proof:** For any $(x,y) \in X \times X$, we have:

$(\mu \times v)(0,0) = \min\{\mu(0), v(0)\}$

$\geq \min\{\mu(x), v(y)\} = (\mu \times v)(x,y)$.

Let $(x_1, x_2)$, $(y_1, y_2)$ and $(z_1, z_2) \in X \times X$.

$(\mu \times v)((x_1, x_2) * (z_1, z_2)) = (\mu \times v)(x_1 * z_1, x_2 * z_2)$

$= \min\{\mu(x_1 * z_1), v(x_2 * z_2)\}$

$\geq \min\{\min\{\mu((x_1 * y_1) * z_1), \mu(y_1)\}, \min\{v((x_2 * y_2) * z_2), v(y_2)\}\}$

$= \min\{\min\{\mu((x_1 * y_1) * z_1), v((x_2 * y_2) * z_2)\}, \min\{\mu(y_1), v(y_2)\}\}$

$= \min\{(\mu \times v)((x_1 * y_1) * z_1, (x_2 * y_2) * z_2), (\mu \times v)(y_1, y_2)\}$

$= \min\{(\mu \times v)((x_1 * y_1, x_2 * y_2) * (z_1, z_2)), (\mu \times v)(y_1, y_2)\}$

$= \min\{(\mu \times v)(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)), (\mu \times v)(y_1, y_2)\}$

Hence μ × v is a fuzzy T-ideal of a TM-algebra in X×X.

**Theorem 23:** Let μ and v be fuzzy sets in a TM-algebra X such that μ ×v is a fuzzy T-ideal of X×X. Then:

- (i) Either $\mu(0) \geq \mu(x)$ or $v(0) \geq v(x)$ for all $x \in X$
- (ii) If $\mu(0) \geq \mu(x)$ for all $x \in X$, then either $v(0) \geq \mu(x)$ or $v(0) \geq v(x)$
- (iii) If $v(0) \geq v(x)$ for all $x \in X$, then either $\mu(0) \geq \mu(x)$ or $\mu(0) \geq v(x)$
- (iv) Either μ or v is a fuzzy T-ideal of X.

**Proof:** μ ×v is a fuzzy T-ideal of X×X.

Therefore $(\mu \times v)(0,0) \geq (\mu \times v)(x,y)$ for all $(x,y) \in X \times X$ and $(\mu \times v)((x_1, x_2) * (z_1, z_2)) \geq \min\{(\mu \times v)(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)), (\mu \times v)(y_1, y_2)\}$ for all $(x_1, x_2)$, $(y_1, y_2)$ and $(z_1, z_2) \in X \times X$.

Suppose that $\mu(0) < \mu(x)$ and $v(0) < v(y)$ for some $x, y \in X$.

Then:

$(\mu \times v)(x,y) = \min\{\mu(x), v(y)\}$
$> \min\{\mu(0), v(0)\} = (\mu \times v)(0,0)$,
a contradiction.

Therefore either $\mu(0) \geq \mu(x)$ or $v(0) \geq v(x)$ for all $x \in X$.

Assume that there exist $x, y \in X$ such that:
$v(0) < \mu(x)$ and $v(0) < v(y)$.
Then:
$(\mu \times v)(0,0) = \min\{\mu(0), v(0)\} = v(0)$ and hence
$(\mu \times v)(x,y) = \min\{\mu(x), v(y)\} > v(0) = (\mu \times v)(0,0)$, a contradiction.

Hence if $\mu(0) \geq \mu(x)$ for all $x \in X$, then either:
$v(0) \geq \mu(x)$ or $v(0) \geq v(x)$

Similarly we can prove that if $v(0) \geq v(x)$ for all $x \in X$, then either $\mu(0) \geq \mu(x)$ or $\mu(0) \geq v(x)$.

First we prove that v is a fuzzy T-ideal of X.

Since, by (i), either $\mu(0) \geq \mu(x)$ or $v(0) \geq v(x)$ for all $x \in X$.

Assume that $v(0) \geq v(x)$ for all $x \in X$.

It follows from (iii) that either $\mu(0) \geq \mu(x)$ or $\mu(0) \geq v(x)$.

If $\mu(0) \geq v(x)$ for any $x \in X$, then:

$v(x) = \min\{ \mu(0), v(x) \} = (\mu \times v)(0,x)$.
$v(x*z) = \min\{ \mu(0), v(x*z)\}$
$= (\mu\times v)(0, x*z)$
$= (\mu \times v)(0*0, x*z)$
$= (\mu \times v)((0, x)*(0, z))$
$\geq \min \{(\mu\times v)((0, x)*(0, y))*(0, z)),$
$(\mu \times v)(0, y)\}$
$= \min \{ (\mu\times v)((0*0, x*y)*(0, z)), (\mu\times v)(0, y) \}$
$= \min \{ (\mu\times v)((0*0)*0, (x*y)*z), (\mu \times v)(0, y) \}$
$= \min \{ (\mu \times v)(0, (x*y)*z), (\mu \times v)(0, y) \}$
$= \min \{ v((x*y)*z), v(y) \}$

Hence v is a fuzzy T-ideal of X.
Now we will prove that μ is a fuzzy T-ideal of X.
Let $\mu(0) \geq \mu(x)$.
By (ii) either $v(0) \geq \mu(x)$ or $v(0) \geq v(x)$.
Assume that $v(0) \geq \mu(x)$, Then:

$\mu(x) = \min\{ \mu(x), v(0) \} = (\mu\times v)(x,0)$.

$\mu(x*z) = \min\{ \mu(x*z), v(0) \}$

$= (\mu\times v)(x*z, 0)$

$= (\mu\times v)(x*z, 0*0)$

$= (\mu\times v)((x,0)*(z,0))$

$\geq \min \{(\mu\times v)(((x, 0)*(y, 0))*(z, 0)), (\mu\times v)(y, 0)\}$

$= \min \{(\mu\times v)((x*y, 0*0)*(z, 0)), (\mu\times v)(y, 0)\}$

$= \min \{(\mu\times v)((x*y)*z, 0), (\mu\times v)(y, 0)\}$

$= \min \{\mu((x*y)*z), \mu(y)\}$

Hence μ is a fuzzy T-ideal of X.

**Homomorphism of TM-algebras:**

**Definition 24:** Let X and Y be TM-algebras. A mapping $f : X \rightarrow Y$ is said to be a homomorphism if it satisfies:

$f(x*y) = f(x)*f(y)$, for all $x, y \in X$.

**Definition 25:** Let f: $X \rightarrow X$ be an endomorphism and μ a fuzzy set in X. We define a new fuzzy set in X by $\mu_f$ in X by $\mu_f(x) = \mu(f(x))$ for all x in X.

**Theorem 26:** Let f be an endomorphism of a TM-algebra X. If μ is a fuzzy T-ideal of X, then so is $\mu_f$.

**Proof:** $\mu_f(x) = \mu(f(x)) \leq \mu(0)$

$= \mu(f(0)) = \mu_f(0)$ for all $x \in X$

Let $x, y, z \in X$.

Then:

$\mu_f(x*z) = \mu(f(x*z)) = \mu(f(x)*f(z))$

$\geq \min\{ \mu((f(x)*f(y))*f(z)), \mu(f(y)) \}$

$= \min\{ \mu((f(x*y))*f(z)), \mu(f(y)) \}$

$= \min\{ \mu(f((x*y)*z)), \mu(f(y)) \}$

$= \min\{ \mu_f((x*y)*z), \mu_f(y) \}$.

Hence $\mu_f$ is a fuzzy T-ideal of X.

## DISCUSSION

With minimum conditions in TM-algebra it satisfy these results. In other algebras like BCK/BCI/BCH/ BCC the number of conditions are more.

## CONCLUSION

In this article, we have fuzzified the subalgebra and ideal of TM-algebras into fuzzy subalgebra and fuzzy ideal of TM-algebras. It has been observed that the TM-algebra satisfy the various conditions stated in the BCC/ BCK algebras and can be considered as the generalization of all these algebras. These concepts can further be generalized.

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