

Some Properties of Certain Subclass of Meromorphically Multivalent Functions Defined By Linear Operator

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Abstract: Problem statement: By means of the Hadamard product (or convolution), a linear operator was introduced. This operator was motivated by many authors namely Srivastava, Swaminathan, Owa and many others. The operator was indeed needed to create new ideas in the area of geometric function theory. **Approach:** The linear operator of meromorphic p -valent functions was proposed and defined. The preliminary concept of subordination was introduced to give sharp proofs for certain sufficient conditions of the linear operator aforementioned. **Results:** Having the operator, subordination theorems established by using standard concept of subordination and reduced to well-known results studied by various authors. The operator was then applied for fractional calculus and obtained new subordination theorem. **Conclusion:** Therefore, interesting operators could be obtained with some earlier results and standard methods.

Key words: Hypergeometric, meromorphic, hadamard product, linear operator

INTRODUCTION

Let Σ_p denote the class of meromorphic functions $f(z)$ normalized by:

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+p} z^{n+p} \quad (1)$$

which are analytic and p -valent in the punctured unit disk $U = \{z: 0 < |z| < 1\}$. For $0 \leq \beta < p$, we denote by $S_p^*(\beta)$ and $K_p(p)$ the subclasses of Σ_p consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in U .

The classes $S_p^*(\beta)$, $K_p(p)$ and various other subclasses of Σ_p have been studied rather extensively by (Aouf, 1993; Aouf and Hossen, 1993; Aouf and Srivastava, 1997; Ghanim and Darus, 2009a; 2009b; Joshi and Srivastava, 1999; Kukarni *et al.*, 1998; Mogra, 1990a; 1990b; Owa *et al.*, 1997; Srivastava and Owa, 1992; Uralegaddi and Somanatha, 1992; Yang, 1995).

For functions $f_j(z)$ ($j = 1, 2$) defined by:

$$f_j(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+p,j} z^{n+p} \quad (2)$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by:

$$(f_1 * f_2) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p} \quad (3)$$

Let us define the function $\phi_p(\alpha, \beta; z)$ by:

$$\phi_p(\alpha, \beta; z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} z^{n+1} \quad (4)$$

($\beta \neq 0, -1, -2, \dots$ and $\alpha \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N}$).

where $(\lambda)_n$ is the Pochhammer symbol. We note that:

$$\phi_p(\alpha, \beta; z) = \frac{1}{z^p} {}_2F_1(1, \alpha; \beta; z)$$

Where:

$${}_2F_1(1, \alpha; \beta; z) = \sum_{n=0}^{\infty} \frac{(1)_{n+1} (\alpha)_{n+1}}{(\beta)_{n+1}} \frac{z^n}{n!}$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\phi_p(\alpha, \beta; z)$ using the Hadamard product for $f(z) \in \Sigma_p$, we define a new linear operator $L_p(\alpha, \beta)$ on Σ_p :

$$\begin{aligned} L_p(\alpha, \beta)f(z) &= \phi_p(\alpha, \beta; z) * f(z) \\ &= \frac{1}{z^p} + \sum_{n=0}^{\infty} \left| \frac{(\alpha)_{n+2}}{(\beta)_{n+2}} \right| a_{n+p} z^{n+p} \end{aligned} \quad (5)$$

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The meromorphic functions with the generalized hypergeometric functions were considered recently by (Cho and Kim, 2007; Dziok and Srivastava, 2002; 2003; Liu, 2003; Liu and Srivastava, 2001; 2004a; 2004b).

Also for a function $f(z) \in \Sigma_p$, we define the integral operator $J_{v,p}$ by:

$$J_{v,p} = \frac{v}{z^{v+p}} \int_0^z t^{v+p-1} f(t) dt, \quad v > 0 \tag{6}$$

There are many researches (Uralegaddi and Somanatha, 1991; Whittaker and Watson, 1927; Yang, 1995; 1996) in which the operator $J_{v,p}$ was investigated.

For a function $f \in L_p(\alpha, \beta)f(z)$ we define:

$$I_0(L_p(\alpha, \beta)f(z)) = L_p(\alpha, \beta)f(z)$$

and for $k = 1, 2, 3, \dots$

$$\begin{aligned} I^k(L_p(\alpha, \beta)f(z)) &= z(I^{k-1}L_p(\alpha, \beta)f(z))' + \frac{1+p}{z} \\ &= \frac{1}{z^p} + \sum_{n=0}^{\infty} (n+p)^k \left| \frac{(\alpha)_{n+2}}{(\beta)_{n+2}} \right| a_{n+p} z^{n+p} \end{aligned} \tag{7}$$

We note that $I^k(L_p(\alpha, \beta)f(z))$ studied by Ghanim and Darus (2009a; 2009b).

Also, it follows from (5) that (Liu, 2003):

$$z(L_p(\alpha, \beta)f(z))' = \alpha L_p(\alpha+1, \beta)f(z) - (\alpha+p)L_p(\alpha, \beta)$$

and

$$\begin{aligned} z(I^k L_p(\alpha, \beta)f(z))' \\ = \alpha I^k L_p(\alpha+1, \beta)f(z) - (\alpha+p)I^k L_p(\alpha, \beta) \end{aligned} \tag{8}$$

We denote by $\Sigma_p(k, \gamma, \delta, \mu, \lambda)$ the class of all functions $f(z) \in \Sigma_p$ such that:

$$\begin{aligned} \Re \left\{ (1-\lambda) \left(\frac{I^k L(\alpha, \beta)f(z)}{I^k L(\alpha, \beta)g(z)} \right)^\mu \right. \\ \left. + \lambda \left(\frac{I^k L(\alpha+1, \beta)f(z)}{I^k L(\alpha+1, \beta)g(z)} \right) \cdot \left(\frac{I^k L(\alpha, \beta)f(z)}{I^k L(\alpha, \beta)g(z)} \right)^{\mu-1} \right\} > \gamma \end{aligned} \tag{9}$$

where, $g(z) \in \Sigma_p$ satisfies the following condition:

$$\Re \left\{ \frac{I^k L(\alpha, \beta)g(z)}{I^k L(\alpha+1, \beta)g(z)} \right\} > \delta, \quad (0 \leq \delta < 1, z \in U) \tag{10}$$

where, γ and μ are real numbers such that $0 \leq \gamma < 1, \mu > 0$ and $\lambda \in \mathbb{C}$ with $\Re\{\lambda\} > 0$.

MATERIALS AND METHODS

To establish our main results we need the following lemmas.

Lemma A: Let Ω be a set in the complex plane \mathbb{C} and let the function $\Psi: \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfy the condition $\Psi(ir_2, s_1) \notin \Omega$ for all real $r_2, s_1 \leq \frac{1+r_2^2}{2}$. If $q(z)$ is analytic in U with $q(0) = 1$ and $\Psi(q(z), zq'(z)) \in \Omega, z \in U$ then $\Re\{q(z)\} > 0, (z \in U)$ (Miller and Mocanu, 1978).

Lemma B: If $q(z)$ is analytic in U with $q(0) = 1$ and if $\lambda \in \mathbb{C} \setminus \{0\}$ with $\Re\{\lambda\} > 0$ then $\Re\{q(z) + \lambda zq'(z)\} > \gamma, (0 \leq \gamma < 1)$ implies $\Re\{q(z)\} > \gamma(1-\gamma)(2\xi-1)$, where ξ is given by:

$$\xi = \xi(\Re\lambda) = \int_0^1 (1+t^{\Re\lambda})^{-1} dt$$

which is increasing function of $\Re\{\lambda\}$ and $\frac{1}{2} \leq \xi < 1$.

The estimate is sharp in the sense that the bound cannot be improved (Ponnusamy, 1995).

For real or complex numbers $a, b, c, (c \neq z_0^-)$, the Gauss hypergeometric function is defined by:

$${}_2F_1(\alpha, \beta, \gamma; z) = 1 + \frac{\alpha\beta}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \dots$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in the unit disc U (Whittaker and Watson, 1927).

Each of the identities (asserted by Lemma C) is fairly well known (Liu and Srivastava, 2004b; Whittaker and Watson, 1927).

Lemma C: For real or complex parameters $a, b, c, (c \neq z_0^-)$:

$$\begin{aligned} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt \\ = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} {}_2F_1(\alpha, \beta, \gamma; z) \end{aligned} \tag{11}$$

$(\Re(c) > \Re(b) > 0)$:

$$\begin{aligned}
 {}_2F_1(\alpha, \beta, \gamma; z) &= (1-z)^{-\alpha} {}_2F_1(\alpha, \gamma-\beta, \gamma; \frac{z}{z-1}); \\
 {}_2F_1(\alpha, \beta, \gamma; z) &= {}_2F_1(\beta, \gamma, \gamma; z)
 \end{aligned}
 \tag{12}$$

and

$${}_2F_1(1, 1, 2; \frac{1}{2}) = 2 \ln 2
 \tag{13}$$

RESULTS

Theorem 1: Let $f \in \Sigma_p(k, \gamma, \delta, \mu, \lambda)$, $a \in \mathbb{R} \setminus \{0\}$ and $\lambda \geq 0$. Then:

$$\Re \left(\frac{I^k L(\alpha, \beta) g(z)}{I^k L(\alpha, \beta) f(z)} \right) > \frac{2\alpha\gamma\mu + \delta\lambda}{2\alpha\mu + \delta\lambda}
 \tag{14}$$

$(0 \leq \gamma < 1, \mu > 0, z \in U)$ where the function $g(z) \in \Sigma_p$ satisfies the condition (10).

Proof: Let $\xi = \frac{2\alpha\gamma\mu + \delta\lambda}{2\alpha\mu + \delta\lambda}$ and we define the function $q(z)$ by:

$$q(z) = \frac{1}{(1-\xi)} \left\{ \left(\frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} \right)^\mu - \xi \right\}
 \tag{15}$$

Then $q(z)$ is analytic in U and $q(0) = 1$. If we set:

$$h(z) = \frac{I^k L(\alpha, \beta) g(z)}{I^k L(\alpha + 1, \beta) g(z)}
 \tag{16}$$

then by the hypothesis $\Re\{h(z)\} > \delta$. Differentiating (14) and using the identity (8), we have:

$$\begin{aligned}
 &(1-\lambda) \left(\frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} \right)^\mu \\
 &+ \lambda \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \left(\frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} \right)^{\mu-1} \\
 &= \left[\xi + (1-\xi)q(z) \right] + \frac{\lambda(1-\xi)}{\alpha\mu} h(z) zq'(z)
 \end{aligned}$$

Let us define the function $\Psi(r, s)$ by:

$$\Psi(r, s) = \xi + (1-\xi)r + \frac{\lambda(1-\xi)}{\alpha\mu} h(z)s
 \tag{17}$$

Using (17) and the fact that $f \in \Sigma_p(k, \gamma, \delta, \mu, \lambda)$, we obtain:

$$\{\Psi(q(z), zq'(z)); z \in U\} \subset \Omega = \{w \in \mathbb{C} : \Re(w) > \gamma\}$$

Now for all real $r_2, s_1 \leq \frac{1+r_2^2}{2}$, we have:

$$\begin{aligned}
 \Re\{\Psi(ir_2, s_1)\} &= \xi + \frac{\lambda(1-\xi)}{\alpha\mu} \Re\{h(z)\} \\
 &\leq \xi - \frac{\lambda\delta(1-\xi)(1+r_2^2)}{2\alpha\mu} \leq \xi - \frac{\lambda\delta(1-\xi)}{2\alpha\mu} = \gamma
 \end{aligned}$$

Hence for each $z \in U$, $\Psi(ir_2, s_1) \notin \Omega$. Thus by Lemma A, we have $\Re\{q(z)\} > 0$ and hence:

$$\Re \left(\frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} \right)^\mu > \xi, \quad z \in U$$

This proves Theorem 1.

Corollary 1: Let the functions $f(z)$ and $g(z)$ be in Σ_p and let $g(z)$ satisfy the condition (10). If $a \in \mathbb{R} \setminus \{0\}$, $\lambda \geq 1$ and:

$$\Re \left\{ (1-\lambda) \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} + \lambda \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \right\} > \gamma
 \tag{18}$$

$0 \leq \gamma < 1, z \in U$. Then:

$$\Re \left\{ \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \right\} > \xi = \frac{\gamma(2\alpha + \delta) + \delta(\lambda - 1)}{2\alpha + \lambda\delta}
 \tag{19}$$

Proof: We have:

$$\begin{aligned}
 &\lambda \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \\
 &= \left[(1-\lambda) \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} + \lambda \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \right] \\
 &+ (\lambda - 1) \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)}
 \end{aligned}$$

Since $\lambda > 1$ making use of (18) and (14) (for $\mu = 1$), we deduce that:

$$\Re \left\{ \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \right\} > \xi = \frac{\gamma(2\alpha + \delta) + \delta(\lambda - 1)}{2\alpha + \lambda\delta}$$

Corollary 2: Let $\lambda \in \mathbb{C} \setminus \{0\}$ with $\Re\{\lambda\} > 0$ and $a \in \mathbb{R} \setminus \{0\}$. If $f(z) \in \Sigma_p$ satisfies the following condition:

$$\Re\left\{(1-\lambda)\left(z^p \Gamma^k L(\alpha, \beta) f(z)\right)^\mu + \lambda\left(z^p \Gamma^k L(\alpha+1, \beta) f(z)\right) \cdot \left(z^p \Gamma^k L(\alpha, \beta) f(z)\right)^{\mu-1}\right\} > \gamma$$

$(0 \leq \gamma < 1, \mu > 0, p \in \mathbb{N}, z \in U)$

then

$$\Re\left\{z^p \Gamma^k L(\alpha, \beta) f(z)\right\}^\mu > \frac{2\mu\alpha\gamma + \Re\{\lambda\}}{2\mu\alpha + \Re\{\lambda\}} \tag{20}$$

Further, if $\lambda \geq 1$; $a \in \mathbb{R} \setminus \{0\}$ and $f(z) \in \Sigma_p$ satisfies $\Re\left\{(1-\lambda)z^p \Gamma^k L(\alpha, \beta) f(z) + \lambda\left(z^p \Gamma^k L(\alpha+1, \beta) f(z)\right)\right\} > \gamma$. Then:

$$\Re\left\{z^p \Gamma^k L(\alpha+1, \beta) f(z)\right\} > \frac{(2a+1)\gamma + \lambda - 1}{2a + \lambda}, \tag{21}$$

$(0 \leq \gamma < 1, p \in \mathbb{N}, z \in U)$

Proof: The results (20) and (21) follow by putting $g(z) = \frac{1}{z^p}$ in Theorem 1 and Corollary 1, respectively.

Theorem 2: Let $\lambda \in \mathbb{C} \setminus \{0\}$ with and $a \in \mathbb{R} \setminus \{0\}$. If $f(z) \in \Sigma_p$ satisfies the following condition:

$$\Re\left\{(1-\lambda)\left(z^p \Gamma^k L(\alpha, \beta) f(z)\right)^\mu + \lambda\left(z^p \Gamma^k L(\alpha+1, \beta) f(z)\right) \cdot \left(z^p \Gamma^k L(\alpha, \beta) f(z)\right)^{\mu-1}\right\} > \gamma$$

$(0 \leq \gamma < 1, \mu > 0, p \in \mathbb{N}, z \in U)$

Then:

$$\Re\left\{z^p \Gamma^k L(\alpha, \beta) f(z)\right\}^\mu > \gamma + (1-\gamma)(2\rho - 1) \tag{23}$$

Where:

$$\rho = \frac{1}{2} {}_2F_1\left(1, 1, \frac{\mu a}{\Re\{\lambda\}}, \frac{1}{2}\right), \quad z \in U \tag{24}$$

Proof: Let:

$$q(z) = \left(z^p \Gamma^k L(\alpha, \beta) f(z)\right)^\mu \tag{25}$$

Then $q(z)$ is analytic in U with $q(0) = 1$. Differentiating $q(z)$ with respect to z and using the identity (8), we obtain:

$$(1-\lambda)\left(z^p \Gamma^k L(\alpha, \beta) f(z)\right)^\mu + \lambda\left(z^p \Gamma^k L(\alpha+1, \beta) f(z)\right) \cdot \left(z^p \Gamma^k L(\alpha, \beta) f(z)\right)^{\mu-1} = q(z) + \frac{\lambda z q'(z)}{\mu a}$$

So that by the hypothesis (22), we have:

$$\Re\left\{q(z) + \frac{\lambda z q'(z)}{\mu a}\right\} > \gamma$$

In view of Lemma B, this implies that:

$$\Re\{q(z)\} > \gamma + (1-\gamma)(2\rho - 1)$$

Where:

$$\rho = \rho(\Re\{\lambda\}) = \int_0^1 \left(1 + t^{\frac{\Re\{\lambda\}}{\mu a}}\right)^{-1} dt$$

Putting $\Re\{\lambda\} = \lambda_1 > 0$, we have:

$$\rho = \int_0^1 \left(1 + t^{\frac{\lambda_1}{\mu a}}\right)^{-1} dt = \frac{\mu \alpha}{\lambda_1} \int_0^{\frac{\mu \alpha}{\lambda_1}} u^{\frac{\mu \alpha}{\lambda_1} - 1} (1 + u)^{-1} du$$

Using (11)-(13), we obtain:

$$\rho = {}_2F_1\left(1, \frac{\mu \alpha}{\lambda_1}; \frac{\mu \alpha}{\lambda_1} + 1; -1\right) = \frac{1}{2} {}_2F_1\left(1, 1, \frac{\alpha}{\lambda} + 1, \frac{1}{2}\right)$$

Thus the proof of Theorem 2 is complete.

Corollary 3: Let $\lambda \in \mathbb{R}$, $\mu = 1$ with $\lambda \geq 1$. If $f(z) \in \Sigma_p$ satisfies:

$$\Re\left\{(1-\lambda)\left(z^p \Gamma^k L(\alpha, \beta) f(z)\right) + \lambda\left(z^p \Gamma^k L(\alpha+1, \beta) f(z)\right)\right\} > \gamma \tag{26}$$

$(0 \leq \gamma < 1, \alpha \in \mathbb{R} \setminus \{0\}, p \in \mathbb{N}, z \in U)$, then:

$$\Re\left\{z^p \Gamma^k L(\alpha+1, \beta) f(z)\right\} > \gamma + (1-\gamma)(2\rho^* - 1)(1 - \lambda^{-1})$$

Where:

$$\rho^* = \frac{1}{2} {}_2F_1\left(1, 1, \frac{\alpha}{\lambda} + 1, \frac{1}{2}\right)$$

Proof: The result follows by using the identity:

$$\begin{aligned} & \lambda z^p I^k L(\alpha + 1, \beta) f(z) \\ &= \left[(1 - \lambda) (z^p I^k L(\alpha, \beta) f(z)) + \lambda (z^p I^k L(\alpha + 1, \beta) f(z)) \right] \quad (27) \\ &+ (\lambda - 1) (z^p I^k L(\alpha, \beta) f(z)) \end{aligned}$$

In the following theorem, we shall extend the above results as follows.

Theorem 3: Suppose that the functions $f(z)$ and $g(z)$ are in Σ_p and suppose $g(z)$ satisfies the condition (10). If:

$$\Re \left\{ \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} - \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} \right\} > \frac{(1 - \gamma)\delta}{2\alpha} \quad (28)$$

($0 \leq \gamma < 1, \alpha \in \mathbb{R} \setminus \{0\}, 0 \leq \delta < 1, z \in U$), then

$$\Re \left\{ \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \right\} > \gamma \quad (29)$$

and

$$\Re \left\{ \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \right\} > \frac{(2\alpha + 1 + \delta)\gamma - \delta}{2\alpha} \quad (30)$$

Proof: Let:

$$q(z) = \frac{1}{(1 - \gamma)} \left\{ \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} - \gamma \right\} \quad (31)$$

Then $q(z)$ is analytic in U with $q(0) = 1$. Putting:

$$\Phi(z) = \frac{I^k L(\alpha, \beta) g(z)}{I^k L(\alpha + 1, \beta) g(z)} \quad (32)$$

We observe that by hypothesis, $\Re\{\Phi(z)\} > \delta$, in U . A simple computation shows that:

$$\begin{aligned} \frac{(1 - \gamma) z q'(z) \Re\{\Phi\}}{\alpha} &= \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \\ - \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} &= \Psi(q(z), zq'(z)) \end{aligned}$$

Where:

$$\Psi(r, s) = \frac{(1 - \gamma)\Phi(z)s}{\alpha}, \quad a \in \mathbb{R} \setminus \{0\}$$

Using the hypothesis (28), we obtain:

$$\Psi(q(z), zq'(z); z \in U) \subset \Omega = \left\{ w \in \mathbb{C} : \Re(w) > -\frac{\delta(1 - \gamma)}{2\alpha} \right\}$$

Now, for all real $r_2, s_1 \leq -\frac{1 + r_2^2}{2}$, we have:

$$\begin{aligned} \Re\{\Psi(ir_2, s_1)\} &= \frac{s_1(1 - \gamma)\Re\{\Phi\}}{\alpha} \\ &\leq -\frac{\delta(1 - \gamma)(1 - r_2^2)}{2\alpha} \leq -\frac{\delta(1 - \gamma)}{2\alpha} \end{aligned}$$

This shows that $\Re\{\Psi(ir_2, s_1)\} \notin \Omega$ for each $z \in U$. Hence by Lemma A, we have $\Re\{q(z)\} > 0, z \in U$. This proves (32). The proof of (33) follows by using (32) and (33) in the identity:

$$\begin{aligned} & \Re \left\{ \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} \right\} \\ &= \Re \left\{ \frac{I^k L(\alpha + 1, \beta) f(z)}{I^k L(\alpha + 1, \beta) g(z)} - \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} \right\} \\ &+ \Re \left\{ \frac{I^k L(\alpha, \beta) f(z)}{I^k L(\alpha, \beta) g(z)} \right\} \end{aligned}$$

This completes the proof of Theorem 3.

DISCUSSION

By putting some values in Corollaries 2 and 3 and Theorem 3 we can obtain the following:

- Putting $\lambda = 1$ and $\alpha, \beta > 0$ in Corollary 2, we have $\Re\{z^p I^k L(\alpha + 1, \beta) f(z) \cdot (z^p I^k L(\alpha, \beta) f(z))^{\mu - 1}\} > \gamma$ implies $\Re\{z^p I^k L(\alpha, \beta) f(z)\}^\mu > \frac{2\alpha\gamma + \Re\{\lambda\}}{2\alpha + \Re\{\lambda\}}, z \in U$
- For $\lambda \in \mathbb{C} \setminus \{0\}$ with $\Re\{\lambda\} > 0, \mu = 1$ and $\alpha, \beta > 0$ in Corollary 2, we have $\Re\{(1 - \lambda) z^p I^k L(\alpha, \beta) f(z) + \lambda (z^p I^k L(\alpha + 1, \beta) f(z))\} > \gamma, (0 \leq \gamma < 1)$ implies $\Re\{z^p I^k L(\alpha, \beta) f(z)\} > \frac{2\alpha\gamma + \Re\{\lambda\}}{2\alpha + \Re\{\lambda\}}, z \in U$ Choosing $k = 0$ and $\alpha = \beta = p$ appropriately in Corollary 2, we obtain the following results

- For $\lambda = 1, k = 0$ and $\alpha = \beta = p$ in Corollary 2, we have $\Re\left\{\frac{zf'(z)}{pf(z)}(z^pf(z))^\mu\right\} > \gamma, (0 \leq \gamma < 1, \mu > 0, p \in \mathbb{N})$
implies $\Re\{z^pf(z)\}^\mu > \frac{2p\mu+1}{2p\mu+1}, z \in U$
- For $\lambda \in \mathbb{C} \setminus \{0\}$ with $\Re\{\lambda\} > 0, \mu = 1, k = 0$ and $\alpha = \beta = p$ in Corollary 2, we have $\Re\left\{(1-\lambda)z^pf(z) + \frac{\lambda}{p}(z^{p+1}f'(z))\right\} > \gamma, (0 \leq \gamma < 1, \mu > 0, p \in \mathbb{N})$ implies $\Re\{z^pf(z)\} > \frac{2p\gamma + \Re\{\lambda\}}{2p + \Re\{\lambda\}}, z \in U$
- Replacing $f(z)$ by $-\frac{zf'(z)}{p}$ in the result (ii), we have $-\Re\left\{(1-\lambda)\frac{z^{p+1}}{p}f(z) + \frac{\lambda}{p^2}(z^{p+2}f''(z))\right\} > \gamma$ implies $-\Re\left\{\frac{z^{p+1}}{p}f'(z)\right\} > \frac{2p\gamma + \Re\{\lambda\}}{2p + \Re\{\lambda\}}, z \in U$
- For $\lambda \in \mathbb{R}$ with $\lambda \geq 1; \mu = 1, k = 0$ and $\alpha = \beta = p$ in Corollary 2, we have $\Re\left\{(1-\lambda)z^pf(z) + \frac{\lambda}{p}(z^{p+1}f'(z))\right\} > \gamma (0 \leq \gamma < 1, z \in U)$ implies $\Re\{z^pf(z)\} > \frac{(2p+1)\gamma + \lambda - 1}{2p + \lambda}$

Next by choosing values in Corollary 3 we have the following remarks:

- We note that if $\lambda = \mu > 0, \alpha = \beta = p$ and $k = 0$ in Corollary 3, that is:
$$\Re\left\{(1-\lambda)(z^pf(z))^\lambda + \frac{\lambda z^{p+1}}{p}f'(z)(z^pf(z))^{\lambda-1}\right\} > \gamma \quad (33)$$

Then (20) implies that:

$$\Re\{z^pf(z)\}^\lambda > \frac{2p\gamma+1}{2p+1}, (z \in U) \quad (34)$$

whereas if $f(z) \in \Sigma_p$ satisfies the condition (33) then by using Theorem 2, we have:

$$\Re\{z^pf(z)\}^\lambda > 2(1-\ell n_2)\gamma + (2\ell n_2 - 1)$$

which is better than (34)

- We observe that if $\lambda \in \mathbb{R}$ satisfying $\lambda \geq 0$ and $k(z) = \frac{I^k L(\alpha+1, \beta)f(z)}{I^k L(\alpha+1, \beta)g(z)} + \left(\frac{1}{\lambda} - 1\right) \frac{I^k L(\alpha, \beta)f(z)}{I^k L(\alpha, \beta)g(z)}$ then

from Theorem 1 (for $\mu = 1$), we have $\Re\{k(z)\} > \frac{\gamma}{\lambda}$
Implies:

$$\Re\left\{\frac{I^k L(\alpha, \beta)f(z)}{I^k L(\alpha, \beta)g(z)}\right\} > \frac{2\alpha\gamma + \lambda\delta}{2\alpha + \lambda\delta} \quad (35)$$

Whenever:

$$\Re\left\{\frac{I^k L(\alpha, \beta)g(z)}{I^k L(\alpha+1, \beta)g(z)}\right\} > \delta, \quad 0 \leq \delta < 1$$

Let $\lambda \rightarrow +\infty$, then from (35), we have $\Re\{k(z)\} > 0$
implies:

$$\Re\left\{\frac{I^k L(\alpha, \beta)f(z)}{I^k L(\alpha+1, \beta)g(z)}\right\} \geq 1, z \in U$$

Whenever:

$$\Re\left\{\frac{I^k L(\alpha, \beta)g(z)}{I^k L(\alpha+1, \beta)g(z)}\right\} > \delta, (0 \leq \delta < 1, z \in U)$$

Finally, we have the following:

- Putting $\alpha = \beta = p, k = 0$ and $g(z) = \frac{1}{z^p}$ in Theorem 3 for all $z \in U$, we obtain:

$$\Re\left\{z^pf(z) + \frac{z^{p+1}}{p}f'(z)\right\} > -\frac{\delta(1-\gamma)}{2p}$$

implies $\Re\{z^pf(z)\} > \gamma$ and:

$$\Re\left\{2z^pf(z) + \frac{z^{p+1}}{p}f'(z)\right\} > \frac{(2p+\delta)\gamma - \delta}{2p}$$

- For $\alpha = p+1, \beta = p, k = 0$ and $g(z) = \frac{1}{z^p}$ in Theorem 3 for all $z \in U$, we have $\Re\{z^p I^k L(p+2, p)f(z) - z^p I^k L(p+1, p)f(z)\} > -\frac{(1-\gamma)\delta}{2(p+1)}$ ($0 \leq \gamma < 1, 0 \leq \delta < 1$), implies $\Re\{z^p I^k L(p+1, p)f(z)\} > \gamma$ and $\Re\{z^p I^k L(p+2, p)f(z)\} > \frac{(2(p+1)+\delta)\gamma - \delta}{2(p+1)}$

CONCLUSION

The operator defined was motivated by various work studied earlier by the authors (Joshi and Srivastava, 1999; Liu and Srivastava, 2001; 2004a; 2004b). This operator can be generalized further and many other results such as the coefficient estimates and distortion theorem can be obtained.

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