Journal of Mathematics and Statistics 6 (3): 333-341, 2010
ISSN 1549-3644

A Study of Naghdi's Shell with a Unilateral Contact of a Rigid Obstacle

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Abstract: Problem statement: In this study we considered a shell modeled by Naghdi's equations with a unilateral contact of a rigid body. **Approach:** This model has been studied by Blouza and Le Dret (1994b) but without contact. **Results:** In this study, we studied the existence, uniqueness and continuity of the deformation of this shell with respect to the data. **Conclusion:** We proposed to approximate the model by a finite element method.

Key words: Shells, finite element methods

INTRODUCTION

Several problems in mechanics, physics, control and those dealing with contacts, lead to the study of systems of variational inequalities (Ciarlet, 1978; Grisvard, 1985; Haslinger *et al*., 1996).

A shell is a tridimensional continuous media, where one of the dimensions, namely the thickness e, is relatively small with respect to the others, i.e. the length, the width and the radius of the curvature of the medium surface of the shell.

An example of a closed shell is the one formed by a plane and a part of a cylinder. According to the size of the thickness e, a shell is said to be thick or thin. Under the action of charges that are sufficiently small (in the linear case), the shell gets a deformation according to physical laws such as tridimensional elasticity.

There are two different families of linear thin elastic shell models: one due to Reissner that is based on the Cosserat surface theory (Bernadou, 1994) for details). This has been developed by Naghdi in 1963, in the sense of taking in account the effect of a transversal cut. The second one is based on Kircchoff-Love theory (Blouza and Le Dret, 1994a). It has been developed by Koiter in 1970, where he suggested a bi-dimensional model for thin linearly elastic shells, where the unknowns are the displacement field of the surface points, with neglected cut effects (Lions and Stampacchia, 1967; Slimane *et al*., 2002).

Many authors such as (Bernadou, 1994; Bernadou *et al*., 1994; Blouza and Le Dret, 1994a; Blouza *et al*., 2006) have mainly studied the existence and uniqueness of the solutions for Naghdi's model as an academic example. Here we focus on the same model with a unilateral contact, and this is a particular case in real applications such as cars, boats and plane wings modeling.

Naghdi's model: Let S be a surface of R^3 defined by $S = \varphi(\overline{\Omega})$ with $\varphi : \Omega \to R^3$, $\Omega \subset R^2$. $\forall x \in \Omega, \varphi(x)$ is a generic point of S.

Definition of a shell: A non deformed shell of a surface S of thickness e is a set of R^3 given by:

$$\overline{C} = \left\{ \varphi(x, y) = \phi(x) + z a_3(x), x \in \overline{\Omega} \right\}$$
$$\text{and } -\frac{1}{2} e(x) \le z \le \frac{1}{2} e(x)$$

The thickness of the shell is defined by the application $e : \Omega \to R_+^*$. We have: $\phi \in C^1(\Omega)$:

$$a_1(x) = \frac{\partial \phi(x)}{\partial x_1} = \partial_1 \phi(x)$$

and

$$a_2(x) = \frac{\partial \phi(x)}{\partial x_2} = \partial_2 \phi(x)$$

We suppose that the vectors $a_1(x)$ and $a_2(x)$ are linearly independent in each point x of S. We define the unitary normal vector by:

$$a_3(x) = \frac{a_1(x) \wedge a_2(x)}{\left| a_1(x) \wedge a_2(x) \right|}$$

at the surface point $\varphi(x)$.

Definition of a deformed shell: By taking in account the transversal cut implies that the normal vector can be

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deformed, i.e., there is a rotation. Therefore the normal vector $a_3(x)$ is transformed into a vector $a_3^*(x)$ and by neglecting the effects of pinching and considering the constraints to be approximated in the plane. The deformed shell C^* is described by the points:

$$\overline{C^*} = \left\{\varphi^*(x,z) = \phi^*(x) + za_3^*(x), x \in \overline{\Omega}\right\}$$

and

$$-\frac{1}{2}e(x) \le z \le \frac{1}{2}e(x)$$

$\overline{C^*}$ is the closure of C^*.

Then the distance between a point of the shell and the middle surface remains constant during the deformation process. Let $\phi^*(x)$ be a card deformation of the middle surface defined by:

$$\phi^*(x) = \phi(x) + u(x) \text{ and } a_3^*(x) = a_3(x) + r$$

r is the rotational transversal vector. This is a field vector that measures a linear variation of the normal vector $a_3(x)$ of a surface.

The parameters r_α are the linear components with respect to the local basis of the normal vector of rotation $a_3(x)$ field, where r is given by:

$$r = a_3^*(x) - a_3(x),\ r = r_\alpha a^\alpha \text{ and } r.a_3 = 0$$

i.e., r does not have a normal component.

Displacement of a shell point: The displacement of a point U(x,z) is written as:

$$\begin{aligned} U(x,z) &= \varphi^*(x,z) - \varphi(x,z) \\ &= \phi^*(x) + za_3^*(x) - (\phi(x) + za_3(x)) \\ &= \phi(x) + u(x) + z(a_3(x) + r_\alpha a^\alpha) \\ &\quad - (\phi(x) + za_3(x)) \\ &= u(x) + zr_\alpha a^\alpha \end{aligned}$$

This applies that:

$$U(x,z) = u(x) + zr(x)$$

Therefore $U(x,z)$ is the displacement of the point $\phi(x)$ of S and r is the normal vector of rotation $a_3(x)$.

Remark 1: With these hypotheses, the displacement $u(x)$ and the rotation r(x) of the normal vector at the point $\phi(x)$ of the middle surface, we can define the displacement of the point $\varphi(x,z)$ of the shell by:

$$U(x,z) = u(x) + zr(x)$$

The main unknowns in Naghdi's model are the displacement $u(x)$ of the surface points $\phi(x)$ in S and the normal vector of the rotation $r(x)$.

Remark 2: For the Koiter's model this is given by:

$$U(x,z) = u(x) + z(\partial_\alpha u.a_3)a^\alpha$$

where, the only unknown is $u(x)$ (the displacement of the point $\phi(x)$ of the medium surface).

Classical formulation: For the unknowns:

$$u = u_i a^i \text{ and } r = r_\alpha a^\alpha$$

with $i = 1,2,3$ and $\alpha = 1,2$. We pose:

$$V_0 = \left\{(v_, , s_\alpha) \in \left[H^1(\Omega)\right]^5, v_i = s_\alpha = 0 \text{ on } \Gamma_0\right\}$$

with Γ_0 is a part of the boundary of $\Omega \subset R^2$.

The formulation of Naghdi's problem in the local basis is given by the following:

Find:

$$(u,r) = ((u_i),(r_\alpha)) \in V_0$$

Such that:

$$a((u,r),(v,s)) = l(v,s),\ \forall (v,s) \in V_0$$

$$\begin{aligned} a((u,r),(v,s)) &= \int_\Omega e a^{\alpha\beta\rho\sigma} \begin{pmatrix} \gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) \\ +\frac{e^2}{12}\chi_{\alpha\beta}(u,r)\chi_{\rho\sigma}(v,s) \end{pmatrix} \sqrt{a}dx \\ &+ \int_\Omega 4e\mu a^{\alpha\beta}\delta_{\alpha 3}(u,r)\delta_{\beta 3}(v,s)\sqrt{a}dx \end{aligned}$$

$$l(v,s) = \int_\Omega pv\sqrt{a}dx \qquad (1)$$

$$a^{\alpha\beta\rho\sigma} = \frac{2\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\rho\sigma} + \mu\left(a^{\alpha\rho} a^{\beta\sigma} + a^{\alpha\sigma} a^{\beta\rho}\right)$$

$a^{\alpha\beta\rho\sigma}$ is the elasticity tensor, λ and μ are The constants of Lamé $\lambda \geq 0$ and $\mu > 0$.

In the case of an homogeneous, isotropic material with Young's modulus $E > 0$ and Poisson's ratio υ, $0 \leq \upsilon < \frac{1}{2}$, the contravariant components of the elasticity tensor $a^{\alpha\beta\rho\sigma}$ are given by:

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\upsilon)}\left(a^{\alpha\rho} a^{\beta\sigma} + a^{\alpha\sigma} a^{\beta\rho}\right) + \frac{E\upsilon}{1-\upsilon^2} a^{\alpha\beta} a^{\rho\sigma}$$

Let us denote by ds the area element. Therefore we write $ds = \sqrt{a}dx$. The tensors γ, χ and δ are the metric, the curvature and the transversal deformation respectively. The tensor of the transversal deformation cut are given by the components:

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}\left(\partial_\alpha u.a_\beta + \partial\ u.a_\alpha\right)$$

$$\chi_{\alpha\beta}(u,r) = \frac{1}{2}\begin{pmatrix} \partial_\alpha u.\partial_\beta a_3 + \partial_\beta u.\partial_\alpha a_3 \\ +\partial_\alpha r.a_\beta + \partial_\beta r.a_\alpha \end{pmatrix}$$

$$\delta_{\alpha 3}(u,r) = \frac{1}{2}\left(\partial_\alpha u.a_3 + r.a_\alpha\right)$$

Lemma 1: Let $u \in H^1(\Omega, R^3), r \in H^1(\Omega, R^3)$ such that $r.a_3 = 0$ and $\phi \in W^{2,\infty}(\Omega, R^3)$ then:

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}\left(\partial_\alpha u.a_\beta + \partial\ u.a_\alpha\right)$$

are defined to be functions of $L^2(\Omega)$ that coincide with the deformed tensor components when u and φ are functions of $C^2(\Omega, R^3)$. We write:

$$\chi_{\alpha\beta}(u,r) = \frac{1}{2}\begin{pmatrix} \partial_\alpha u.\partial_\beta a_3 + \partial_\beta u.\partial_\alpha a_3 \\ +\partial_\alpha r.a_\beta + \partial_\beta r.a_\alpha \end{pmatrix}$$

are defined to be functions of $L^2(\Omega)$ that coincide with the tensor of change in curvature when u, r and φ are functions of $C^2(\Omega, R^3)$. We write:

$$\delta_{\alpha 3}(u,r) = \frac{1}{2}\left(\partial_\alpha u.a_3 + r.a_\alpha\right)$$

are defined to be functions of $L^2(\Omega)$ that coincide with the tensor of the cut when u and r are functions of $C^2(\Omega, R^3)$.

This problem is well posed (Bernadou *et al*., 1994).

Formulation in Cartesian components: We consider a middle surface of shell S. The card ϕ is assumed to be ($\phi \in W^{2,\infty}(\Omega, R^3)$).

It has to be noted that $W^{2,\infty}(\Omega) \to C^1(\Omega)$ (with continuous injection). We introduce the space:

$$V = \left\{(v,s) \in \left[H^1(\Omega, R^3)\right]^2, sa_3 = 0 \text{ in } \Omega, v = s = 0 \text{ on } \Gamma_0\right\}$$

With the norm:

$$\|(v,s)\| = \left(\|v\|^2_{H^1(\Omega,R^3)} + \|s\|^2_{H^1(\Omega,R^3)}\right)^{\frac{1}{2}} \tag{2}$$

Theorem 1: Let $u \in H^1(\Omega, R^3), r \in H^1(\Omega, R^3)$ such that $r.a_3 = 0$, be a displacement and a normal rotation of $a_3(x)$ of the middle surface respectively.

We suppose that $\phi \in W^{2,\infty}(\Omega, R^3)$:

- If we assume that $\gamma_{\alpha\beta}(u) = 0$, then there exists a unique $\psi \in L^2(\Omega, R^3)$ such that:

$$\partial_\alpha u = \psi \wedge \partial_\alpha \varphi$$

- If $\delta_{\alpha 3}(u,r) = 0$ then $\partial_\alpha u.a_3 = -r.a_\alpha$ in $H^1(\Omega)$ Moreover:

$$r.a_\alpha = \varepsilon_{\alpha\beta}\psi.a^\beta s$$

- If $\chi_{\alpha\beta}(u,r) = 0$, then ψs is a vector of R^3 and we have:

$$u(x) = c + \psi \wedge \varphi(x)$$

where c is a constant of R^3 and $r(x)$ is given by:

$$r(x) = -\left(\varepsilon_{\alpha\beta}(x).a^\beta(x)\right).a^\alpha(x)$$

Existence and uniqueness of the solution for Naghdi's model: In this study we introduce a theorem of existence and uniqueness of the solution of the linear Naghdi's model for shells with a middle surface only of a class $W^{2,\infty}$.

We consider a middle surface S of the shell of thickness e and of constants of Lame $\lambda \geq 0$ and $\mu > 0$. If we assume that a shell is clamped in the boundary $\Gamma_0 = \partial\Omega$, we have the following result:

Theorem 2: Let $p \in L^2(\Omega, R^3)$ be the resulting of force density and e the thickness of the shell. Then there exists a unique solution for the variational problem:

Find $(u,r) \in V$ such that:

$$a((u,r),(v,s)) = l(v,s),\ \forall (v,s) \in V$$

with:

$$a((u,r),(v,s)) \text{and}\, l(v,s) \text{ in } (1)$$

The elements $a^{\alpha\beta\rho\sigma}$ of elasticity tensor satisfy a symmetric property and are uniformly strictly positive. To prove this theorem we make use of the Lax-Milgram lemma (Bernadou, 1994), which is based on proving the continuity, and V-ellipticity of the bilinear form $a((u,r),(v,s))$ and the continuity of $l(v,s)$ in (1).

The crucial tool in this demonstration is the lemma of rigid movement.

To establish the V-ellipticity of the bilinear form $a((u,r),(v,s))$ we make use of the following lemma.

Lemma 2: These exist a constant C>0 such that:

$$a((u,r),(v,s)) \geq C \left\{ \begin{array}{l} \sum_{\alpha\beta} \left\| \gamma_{\alpha\beta}(v) \right\|^2_{L^2(\Omega)} \\ + \sum_{\alpha\beta} \left\| \chi_{\alpha\beta}(v) \right\|^2_{L^2(\Omega)} \\ + \sum_{\alpha} \left\| \delta_{\alpha 3}(v,s) \right\|^2_{L^2(\Omega)} \end{array} \right\}^{\frac{1}{2}}$$

$$\forall (v,s) \in \left[H^1(\Omega, R^3) \right]^2$$

Mixed formulation of Naghdi's model: In this formulation, the rotation vector is tangent to a middle surface. With this vector constraint the implementation of conforming discontinuous finite element methods is not possible. This is because of the fact that, in classical formulation, $r_3 = 0$, i.e., $r_i a_{3,i} = 0$ in Ω.

Therefore, we introduce a penalized version of Naghdi's model to approximate the tangent vector of r:

$$a((u_p, r_p),(v,s)) + \frac{1}{p} b(r_p, s) = l(v,s)$$

with (3):

$$b(\lambda, s) = \int_\Omega \partial_\alpha(\lambda.a)\partial_\alpha(s.a_3)\sqrt{a}dx$$

The existence, uniqueness and the convergence of the solution of the penalized model when the parameter of penalization converges to 0, are established in (Blouza, and Le Dret, 1994).

Remark 3:

$$r = r.a_\alpha a^\alpha = r_\alpha a^\alpha = r_1 e_1 + r_2 e_2 + r_3 e_3$$

$r \in H^1(\Omega, R^3)$ do not ensure $ra_3 = 0$. Therefore we assumed this condition to be fulfilled and introduced the space of relaxed functions (without orthogonal constraint at r) to be defined by:

$$V_0 = \left\{ (v,s) \in \left[H^1(\Omega, R^3) \right]^2, v = s = 0 \text{ on } \Gamma_0 \right\}$$

with the norm H^1 in (2).

Theorem 3: Let $p \in \mathbb{R}$ such that $0 < p \leq 1$ and $f \in L^2(\Omega, R^3)$, then there exists a unique solution of problem (3).

Lemma 3: The bilinear form is V_0-elliptic, uniformly for $0 < p \leq 1$.

Remark 4: Note that this formulation allowed us to approximate the constraint $r.a_3 = 0$ when the penalization parameter tends to zero.

Mixed stability formulation: The mixed Naghdi's problem consists of finding $(u,r) \in V_0$ and a Lagrange indicator $\lambda \in H^1_{\Gamma_0}(\Omega)$ satisfying:

$$\begin{aligned} & a((u,r),(v,s)) + \eta \bar{a}((u,r),(v,s)) + b((v,s),\lambda) \\ & = l(v,s),\ \forall (v,s) \in V_0 \end{aligned}$$

$$b((u,r),\mu) = 0,\ \ \forall \mu \in H^1_{\Gamma_0}(\Omega)$$

with

$$\bar{a}((u,r),(v,s)) = \int_\Omega \partial_\alpha (r.a_3)\partial_\alpha (s.a_3)\sqrt{a}dx$$

and:

$$b((v,s),\lambda) = \int_\Omega \partial_\alpha (s.a_3)\partial_\alpha \lambda\sqrt{a}dx$$

Remark 5: This problem is well posed. This is to say:

- The Primal unknown is the unique solution of $a((u,r),(v,s)) = l(v,s),\ \forall (v,s) \in V_0$
- We know that r does not have the covariante component with respect to a_3, i.e., $ra_3 = 0$
  The Lagrange multiplicator λ insured this tangential character of r
- The penalization parameter μ is introduced for stability reasons.
- This formulation in Cartesian components permit to consider the following statement: the general shell with a middle surface that can admit discontinuous curvature is only for ϕ in $W^{2,\infty}$ only.
  We remind that in a classical approach, the card ϕ is of class C^3

Formulation of the contact problem: Here we consider a Naghdi's shell occupying an open bounded domain Ω of a sufficiently regular boundary $\Gamma = \partial\Omega$.

The shell is supposed to have:

- A density on the volume, of force P in Ω
- Homogenous boundary conditions on Γ
- Unilateral contact with a rigid obstacle of equation $x_3 = 0$ on contact surface $\Omega_c = \Omega \setminus \Gamma$

The non deformed shell of middle surface S and with thickness e is a set of R^3 given by:

$$\overline{\Omega} = \{\varphi(x,y) = \phi(x) + za_3(x)\}$$

and:

$$-\frac{1}{2}e(x) \le z \le \frac{1}{2}e(x)$$

The deformed shell is given by the points:

$$\overline{\Omega^*} = \{\varphi^*(x,z) = \varphi^*(x) + f(z)a_3^*(x)\}$$

Where:

f(z) = zs is the distance between a point of the shell where the middle surface remains at the deforming process

$\phi^*(x) = \varphi(x) + u(x)$ with $u(x)$ is a displacement of a point $\varphi(x)$ of the middle surface

$r(x)$ = The rotation normal vector $a_3(x)$

The displacement is given by:

$$(\phi(x) + u(x) + z(a_3(x) + r(x))).e_3 \ge 0, \text{ in } \Omega$$

$z = -\frac{e(x)}{2}$ at the contact area then:

$$\left(\phi(x) + u(x) - \frac{e(x)}{2}(a_3(x) + r(x))\right).e_3 \ge 0, \text{ in } \Omega$$

$$\Leftrightarrow \left(u(x) - \frac{e(x)}{2}r(x).e_3 \ge \left(-\phi(x) + \frac{e(x)}{2}a_3(x)\right)\right).e_3, \text{ in } \Omega$$

We assume that:

$$\varphi(x) = \left(-\phi(x) + \frac{e(x)}{2}a_3(x)\right).e_3$$

We denote by η the reaction of the obstacle on the shell. The relations leading to a unilateral contact (without friction) are given by:

$$\begin{cases} u(x) - \frac{e(x)}{2}r(x).e_3 - \varphi(x) \ge 0, \text{ in } \Omega \\ \eta \ge 0, \text{ in } \Omega \\ \left(u(x) - \frac{e(x)}{2}r(x).e_3 - \varphi(x)\right)\eta = 0, \text{ in } \Omega \end{cases}$$

We use the space $H_0^1(\Omega, R^3)$ of functions in $H^1(\Omega, R^3)$ equals to zero on Γ and we denote by $H^{-1}(\Omega, R^3)$ the space of duality.

Let us introduce the convex subspace K for the authorized displacements, to be defined as:

$$K = \left\{(v,s) \in \left(H_0^1(\Omega, R^3)\right)^2, \left(v - \frac{e}{2}s\right).e_3 \ge \varphi \text{ in } \Omega\right\}$$

and let convex subspace K^* of the distributions χ in $H^{-1}(\Omega, R^3)$, to be:

$$K^* = \{\chi \in H^{-1}(\Omega, R^3), \langle \chi, (v,s) \rangle \geq 0, \forall (v,s) \in K\}$$

with $\langle , \rangle$ the inner product of duality between:

$$H^{-1}(\Omega, R^3) \text{ and } H_0^1(\Omega, R^3)$$

We consider the following variationnal formulation Find:

$$((u,r),\lambda,\eta) \in \left(H_0^1(\Omega,R^3)\right)^2 \times H_0^1(\Omega) \times H^{-1}(\Omega)$$

Such that:

$$(P_e)\begin{cases} a((u,r),(v,s)) + b((v,s),\lambda) - c((\eta,(v,s))) \\ \quad = l(v,s), \quad \forall (v,s) \in \left(H_0^1(\Omega,R^3)\right)^2 \\ b((v,s),\mu) = 0, \quad \forall \mu \in H_0^1(\Omega) \\ \langle \chi - \eta, (u,r) \rangle \geq 0, \quad \forall \chi \in K^* \end{cases}$$

with:

$$a((u,r),(v,s)) = \int_\Omega e a^{\alpha\beta\rho\sigma} \begin{pmatrix} \gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) \\ +\frac{e^2}{12}\chi_{\alpha\beta}(u,r)\chi_{\rho\sigma}(v,s) \end{pmatrix} \sqrt{a}dx + \int_\Omega 4e\mu a^{\alpha\beta}\delta_{\alpha 3}(u,r)\delta_{\beta 3}(v,s)\sqrt{a}dx$$

$$b((v,s),\lambda) = \int_\Omega \partial_\alpha(s.a_3)\partial_\alpha \lambda dx$$

$$c(\eta,(v,s)) = \int_\Omega \eta v dx$$

$$l(v,s) = \int_\Omega P v \sqrt{a} dx$$

and the reduced problem becomes:

Find $((u,r),\lambda) \in K \times H_0^1(\Omega)$ such that:

$$(P_I)\begin{cases} a((u,r),(v,s)-(u,r)) + b((v,s)-(u,r),\lambda) \geq \\ \quad l((v,s)-(u,r)) \\ b((v,s),\mu) = 0, \quad \forall \mu \in H_0^1(\Omega) \end{cases}$$

Theorem 4: For any solution $((u,r),\lambda,\eta)$ of problem (P_e), $((u,r),\lambda)$ is a solution of problem (P_I).

Proof: Let $((u,r),\lambda,\eta)$ be a solution of problem (P_e) and $(u,r) \in K$, $\forall (v,s) \in K$ and by the definition of K^* we have:

$$\langle \eta, (v,s) \rangle \geq 0 \Leftrightarrow -\langle \eta, (v,s) \rangle \leq 0$$

Line three of problem (P_e) leads to:

$$\langle \chi - \eta, (u,r) \rangle \geq 0, \ \forall \chi \in K^*$$

We assume that $\chi = 0$:

$$-\langle \eta, (u,r) \rangle \geq 0 \Leftrightarrow \langle \eta, (u,r) \rangle \leq 0$$

by replacing (v,s) by $(v,s)-(u,r)$ in line one of problem (P_e), we get:

$$a((u,r),(v,s)-(u,r)) + b((v,s)-(u,r),\lambda) \\ -c(\eta,(v,s)-(u,r)) = l((v,s)-(u,r))$$

where:

$$-c(\eta,(v,s)-(u,r)) = -\langle \eta,(v,s)-(u,r) \rangle \\ = -\langle \eta,(v,s) \rangle + \langle \eta,(u,r) \rangle \leq 0 \\ \Rightarrow a((u,r),(v,s)-(u,r)) + b((v,s)-(u,r),\lambda) \geq \\ l((v,s)-(u,r)), \ \forall (v,s) \in K$$

Let $((u,r),\lambda)$ be a solution of problem (P_I) then $((u,r),\lambda,\eta)$ is a solution of (P_e):

$$a((u,r),(v,s)-(u,r)) + b((v,s)-(u,r),\lambda) - \\ l((v,s)-(u,r)) \geq 0, \ \forall (v,s) \in K$$

by using Green's formula, we get:

$$a((u,r),(v,s)-(u,r)) + b((v,s)-(u,r),\lambda) - \\ \langle \eta,(v,s)-(u,r) \rangle - l((v,s)-(u,r)) \geq 0$$

We assume that $(v,s) = (u,r) \pm \varphi$, with $\phi \in D(\Omega, R^3)$,(i.e., φ is of a compact support), then the integral on the contour is zero:

$$a((u,r),\varphi) + b(\varphi,\lambda) = l(\varphi), \ \forall \varphi$$

The integral on a contact area leads to:

$$\langle \eta,(v,s)-(u,r)\rangle \geq 0,\ \forall (v,s)\in K$$

By assuming that:

$$\begin{cases}(v,s)=(0,0)\\(v,s)=2(u,r)\end{cases} \Rightarrow \langle \eta,(u,r)\rangle = 0$$

and with the property of convexity of K^*, we get:

$$\langle \chi-\eta,(u,r)\rangle = 0$$
$$\langle \chi,(u,r)\rangle - \langle \eta,(u,r)\rangle = \langle \chi,(u,r)\rangle \geq 0$$

Theorem 5: For any $P\in H^{-1}(\Omega,R^3)$, the problem (P_e) has a unique solution:

$$((u,r),\lambda,\eta)\in \left(H_0^1(\Omega,R^3)\right)^2\times H_0^1(\Omega)\times H^{-1}(\Omega)$$

Proof: The existence of the solution $((u,r),\lambda)$ of problem is a direct application of Lions-Stampacchia Theorem (Lions and Stampacchia, 1967).

Let us consider:

$$L(v,s)=a((u,r),(v,s))+b((v,s),\lambda)-l(v,s)$$

Remark: In problem (P_I), we have:

- if $(v,s)=(0,0)$, then:

$$-a((u,r),(u,r))-b((u,r),\lambda)\geq -l(u,r)$$

- if $(v,s)=2(u,r)$, then:

$$a((u,r),(u,r))+b((u,r),\lambda)\geq -l(u,r)$$
$$\Rightarrow \quad L(u,r)=0$$

The Ker of the form $\langle\eta(v,s)\rangle$ is characterized by:

$$V=\left\{\begin{matrix}(v,s)\in\left(H_0^1(\Omega,R^3)\right)^2,\left(v-\frac{e}{2}s\right).e_3-\varphi=0,\\ \text{in}\,\Omega\end{matrix}\right\}$$

Let $(v,s)\in V$, then (v,s) and $-(v,s)$ are in K from the problem (P_I) and $L(u,r)=0$, we have:

$$a((u,r),(v,s))-a((u,r),(u,r))+b((v,s),\lambda)$$
$$-b((u,r),\lambda)-l(v,s)+l(u,r)\geq 0$$
$$\Leftrightarrow\ a((u,r),(v,s))+b((v,s),\lambda)-l(v,s)$$
$$-a((u,r),(u,r))+b((v,s),\lambda)-l(u,r)\geq 0$$
$$\Leftrightarrow\ a((u,r),(v,s))+b((v,s),\lambda)-l(v,s)\geq 0$$
$$\Rightarrow \quad L(u,r)=0$$

We replace (v,s) by $-(v,s)$ in $L(u,r)$ to get:

$$a((u,r),(v,s))+b(-(v,s),\lambda)-l(-(v,s))\geq 0$$
$$\Leftrightarrow\ -a((u,r),(v,s))-b((v,s),\lambda)+l(v,s)\geq 0$$
$$\Leftrightarrow\ a((u,r),(v,s))+b((v,s),\lambda)-l(v,s)\leq 0$$
$$\Leftrightarrow\ L(u,r)=0$$

L is of a compact support in V and from the following inf-sup condition:

$$\sup\frac{\langle\eta,(v,s)\rangle}{\|(v,s)\|}\geq\beta\|\eta\|H^{-1}$$

We can prove that there exists $\eta\in H^{-1}(\Omega)$ such that:

Then $((u,r),\lambda,\eta)$ satisfies line one of problem (P_e). The definition of K^* and $L(u,r)=0$, leads to:

$$\langle\chi-\eta,(u,r)\rangle=\langle\chi,(u,r)\rangle-\langle\eta,(u,r)\rangle$$
$$=\langle\chi,(u,r)\rangle\geq 0,\ \forall\chi\in K$$

This proves the existence of the solution.

Let (U_1,λ_1) and (U_2,λ_2) be two solutions of problem (P_I). With $U_1=(u_1,r_1)$ and $U_2=(u_2,r_2)$ then:

$$a(U,W-U)+b(W-U,\lambda)\geq l(W-U),\ \forall W\in K$$
$$a(U_2,-U_2)+b(W-U_2,\lambda_2)\geq l(W-U_2),\forall W\in K$$

$$b(U_1,\lambda_1-\lambda_2)=0s$$
$$b(U_2,\lambda_1-\lambda_2)=0$$
$$b(U_1-U_2,\lambda_1-\lambda_2)=0$$

By adding that $W=U_2$ and $W=U_1$ we have:

$$a(U_1,U_2-U_1)+b(U_2-U_1,\lambda_1)\geq l(U_2-U_1)$$
$$a(U_2,U_1-U_2)+b(U_1-U_2,\lambda_2)\geq l(U_1-U_2)$$

$$\Leftrightarrow \quad \begin{aligned}&a(U_1,U_1-U_2)+b(U_1-U_2,\lambda_1)\geq l(U_1-U_2)\\&a(U_2,U_1-U_2)+b(U_1-U_2,\lambda_2)\geq l(U_1-U_2)\end{aligned}$$

$$a(U_1-U_2,U_1-U_2)+b(U_1-U_2,\lambda_1)\leq 0$$

$$\Leftrightarrow \quad a(U_1-U_2,U_1-U_2)\leq 0$$

$$\Leftrightarrow \quad \alpha\|U_1-U_2\|^2\leq 0 \quad \Leftrightarrow \; U_1=U_2$$

$\lambda_1=\lambda_2$ by the inf-sup condition of $b(\chi,(v,s))$ line three of problem (P_e) gives us:

$$\forall(v,s)\in\left(H_0^1(\Omega,R^3)\right)^2,\langle\eta_1,(v,s)\rangle=\langle\eta_2,(v,s)\rangle$$
$$\Leftrightarrow \quad \eta_1=\eta_2$$

The discrete problem: We introduce a discrete subspace V_h of V such that:

$$V_h=\left\{\begin{matrix}(v_h,s_h)\in\left(C(\Omega,R^3)\right)^2,(v_h,s_h)\in\left(P_1(k)\right)^2\\ v_h=s_h=0,\quad \text{on } \partial\Omega\end{matrix}\right\}$$

and $\dim V_h<\infty$, therefore there exists a basis: $\{\omega_i\}$, $i=1$ to N_h, we can then write:

$$v_h=\sum_{i=1}^{N_h}\beta_i\omega_i$$

$$s_h=\sum_{i=1}^{N_h}\alpha_i\omega_i$$

Now, let us construct a closed convex subset K_h of V_h such that K_h should be reduced to a finite number of constraints on the β_i and α_i:

$$K_h=\left\{\begin{matrix}(v_h,s_h)\in V_h,\ \left(v-\frac{e}{2}s\right).e_3\geq\varphi\\ \text{at every vertex of each triangle } K\end{matrix}\right\}$$

Then $K_h\subset K$ and $K_h\subset V_h$.

We remark that problem (P_l) is equivalent to find $((u_h,r_h),\lambda_h)\in K_h\times M_h$ such that:

$$(P_h)\begin{cases}a((u_h,r_h),(v_h,s_h)-(u_h,r_h))\\ \qquad +b((v_h,s_h)-(u_h,r_h),\lambda_h)\geq\\ l((v_h,s_h)-(u_h,r_h)),\ \forall(v_h,s_h)\in K_h\\ b((v_h,s_h),\mu_h)=0,\ \forall\mu_h\in M_h\end{cases}$$

with:

$$M_h=\{\mu_h=r_h.a_3,\mu_h\in P_0(k),\forall r_h\in V_h\}$$

space of the Lagrange multipliers.

We assume $U=(u,r)$ and $W=(v,s)$.

Theorem 5: Let (U,λ) and (U_h,λ_h) be the solutions of problems (P_l) and (P_h), respectively. Let us denote by $A\in L(V,V')$ the map defined, by $a(U,W)=(AU,W)$, then:

$$\|U-U_h\|_V=\begin{bmatrix}\frac{M^2}{\alpha^2}\|U-W_h\|_V^2+\frac{1}{\alpha}\|P-AU\|_{V'}\\ \|U-W_h\|_V+\|U_h-W\|_V\end{bmatrix}^{\frac{1}{2}}$$

with P is the resultant of the volume force.

Proof: By the definitions of U and W, we have:

$$a(U,U-W)+b(U-W,\lambda)\leq(P,U-W),\ \forall W\in K$$
$$a(U_h,U_h-W_h)+b(U_h-W_h,\lambda)\leq(P,U_h-W_h),$$
$$\forall W_h\in K_h$$

By adding these inequalities and transposing terms, we obtain:

$$a(U,U)+a(U_h,U_h)\leq(P,U-W)+(P,U_h-W_h)$$
$$+a(U,W)+a(U_h,W_h)$$

By subtracting $a(U,U_h)+a(U_h,U)$ from both sides and grouping terms and by using the continuity and the coercively of the bilinear form $a(U,W)$, we deduce:

$$\alpha\|U-U_h\|_V^{\ 2}\leq\begin{bmatrix}\|P-AU\|_{V'}\|U-W_h\|_V\\ +\|P-AU\|_{V'}\|U_h-W\|_V\\ +M\|U-U_h\|_V\|U-W_h\|_V\end{bmatrix}$$

Since:

$$M\|U-U_h\|\|U-W_h\|\leq\frac{M}{\alpha}\|U-W_h\|^2$$

We obtain:

$$\|U - U_h\|_V = \begin{bmatrix} \frac{M^2}{\alpha^2}\|U - W_h\|_V^2 + \frac{1}{\alpha}\|P - AU\|_{V'} \\ \|U - W_h\|_V + \|U_h - W\|_V \end{bmatrix}^{\frac{1}{2}}$$

$$\forall W \in K \text{ and } \forall W_h \in K_h$$

CONCLUSION

By starting with the classical Naghdi's model for a shell in Cartesian coordinates, we derived a model for the contact of this shell with a rigid body. We also proved the well-posedness of the resulting system for the variational inequalities.

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