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Higher-Order Newton-Cotes Formulas

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Abstract: Problem statement: The present work offers equations of Newton-Cotes Integration until twenty segments. **Approach:** It shows Newton-Cotes closed and open integration formulas. The new type Newton-Cotes semi-closed or semi-open was proposed. **Results:** An analysis of error in the technique was made. To estimate the error of integration in discrete data, we propose apply different rules or mix several rules. **Conclusion/Recommendations:** The difference in the result of each formula provides an approximation of the error. Computational routines to generate Newton-Cotes integration rules were presented.

Key words: Numerical methods, numerical integration, quadrature, Newton-cotes, higher-order, analysis of error

INTRODUCTION

The evaluation of integrals, a process known as integration or quadrature, is required in many problems in engineering and science (Sermutlu, 2005). The function f(x), which is to be integrated, may be a known function or a set of discrete data. Some known functions have an exact integral, in which case can be evaluated exactly in closed form. Many known functions, however, do not have an exact integral and an approximate numerical procedure is required to evaluate. In many cases, the function f(x) is known only at a set of discrete points, in which case an approximate numerical procedure is again required to evaluate (Simos, 2008). Numerical integration (quadrature) formulas can be developed by fitting approximating functions (e.g., polynomials) to discrete data and integrating the approximating function:

$$I = \int_{x_1}^{x_N} f(x)dx \cong \int_{x_1}^{x_1+(N-1)h} P(x)dx \quad (1)$$

When the function to be integrated is known at equally spaced points ($\Delta x = h =$ constant) and N is number of points with x ranging x_1, x_1+h, x_1+2h, …, $x_1+(N-1)h$. The polynomial can be fit to the discrete data with much less effort, thus significantly decreasing the amount of effort required (Simos, 2008). The resulting formulas are called Newton-cotes formulas.

The distance between the lower and upper limits of integration is called the range of integration. The distance between any two data points is called an increment ($\Delta x = h$). A linear polynomial requires one increment and two data points to obtain a fit. A quadratic polynomial requires two increments and three data points to obtain a fit. And so on for higher-degree polynomials. The group of increments required to fit a polynomial is called an interval (Kalogiratou and Simos, 2003). A linear polynomial requires an interval consisting of only one increment. A quadratic polynomial requires an interval containing two increments. And so on. The total range of integration can consist of one or more intervals. Each interval consists of one or more increments, depending on the degree of the approximating polynomial.

Closed and open forms of Newton-Cotes formulas are available. The closed forms are those where the data points at the beginning and end of the limits of integration are known. The open forms have integration limits that extend beyond the range of the data (Witteveen *et al*., 2009).

MATERIALS AND METHODS

Newton-cotes closed integration equation: The rule for a single interval is obtained by fitting a first-degree polynomial to two discrete points (Sermutlu and Eyyubog'lu, 2007). The upper limit of integration is $x_2 = x_1+h$, then the integral (I) have the formula:

$$I = \frac{h}{2}(y_1 + y_2) \quad \text{or} \quad I = 0.5hy_1 + 0.5hy_2 \quad \text{Error} \approx O(h^2) \quad (2)$$

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Simpson's 1/3 rule is obtained by fitting a second-degree polynomial to three equally spaced discrete points. The upper limit of integration is x_3, then:

$$I = \frac{h}{3}\left(y_1 + 4y_2 + y_3\right) \quad \text{or} \quad \text{Error} \approx O\left(h^4\right)$$
$$\begin{aligned} I = {} & 0.3333333333333333333333333333333333333333hy_1 \\ & + 1.333333333333333333333333333333333333333hy_2 \\ & + 0.3333333333333333333333333333333333333333hy_3 \end{aligned} \quad (3)$$

Simpson's 3/8 rule is obtained by fitting a third-degree polynomial to four equally spaced discrete points. The upper limit of integration is x_4, then:

$$\begin{aligned} & I = \frac{3h}{8}\left(y_1 + 3y_2 + 3y_3 + y_4\right) \quad \text{or} \quad \text{Error} \approx O\left(h^4\right) \\ & I = 0.375hy_1 + 1.125hy_2 + 1.125hy_3 + 0.375hy_4 \end{aligned} \quad (4)$$

Boole's rule is obtained by fitting a fourth-degree polynomial to five equally spaced discrete points. The upper limit of integration is x_5, then:

$$I = \frac{2h}{45}(7y_1 + 32y_2 + 12y_3 + 32y_4 + 7y_5) \ \text{or Error} \approx O\left(h^6\right)$$
$$\begin{aligned} I = {} & 0.3111111111111111111111111111111111111111hy_1 \\ & + 1.422222222222222222222222222222222222222hy_2 \\ & + 0.5333333333333333333333333333333333333333hy_3 \\ & + 1.422222222222222222222222222222222222222hy_4 \\ & + 0.3111111111111111111111111111111111111111hy_5 \end{aligned} \quad (5)$$

Generally, have the closed formulas, where N is number of points, ci are integer coefficients and cr are real coefficients:

$$\begin{aligned} I = {} & \frac{num}{den}h(ci_1y_1 + ci_2y_2 + ci_3y_3 + ci_4\,y_4 + \ldots \\ & + ci_{N-2}y_{N-2} + ci_{N-1}y_{N-1} + ci_Ny_N) \\ & \text{or} \\ I = {} & h\begin{pmatrix} cr_1\,y_1 + cr_2\,y_2 + cr_3\,y_3 + \ldots \\ + cr_{N-2}\,y_{N-2} + cr_{N-1}\,y_{N-1} + cr_N\,y_N \end{pmatrix} \end{aligned} \quad (6)$$

RESULTS AND DISCUSSION

The Table 1 and 2 shows of coefficients for higher order closed integration formulas. Figure 1 shows the generation of rules in Maple 12.0®.

Newton-cotes open integration equation: In the open integration formulas, the first (y_1) and last (y_N) point does not appear in equation. The midpoints rule for a double interval is obtained by fitting a zero-degree polynomial to three discrete points (Espelid, 2003). The upper limit of integration is $x_3 = x_1+2h$, then the integral (I) have the formula:

$$I = 2h\left(y_2\right) \quad \text{or} \quad I = 2hy_2 \qquad \text{Error} \approx O\left(h^2\right) \qquad (7)$$

For three intervals, the rule is obtained by fitting a first-degree polynomial to four discrete points. The upper limit of integration is $x_4 = x_1+3h$, then the integral (I) have the formula:

$$I = \frac{3h}{2}\left(y_2 + y_3\right) \quad \text{or} \quad I = 1.5hy_2 + 1.5hy_3 \qquad \text{Error} \approx O\left(h^2\right) \qquad (8)$$

For N = 5, the rule is obtained by fitting a second-degree polynomial to four equally spaced discrete points. The upper limit of integration is x_5, then:

$$I = \frac{4h}{3}\left(2y_2 - y_3 + 2y_4\right) \quad \text{or} \quad \text{Error} \approx O\left(h^4\right)$$
$$\begin{aligned} I = {} & 2.666666666666666666666666666666666666667hy_2 \\ & -1.333333333333333333333333333333333333333hy_3 \\ & +2.666666666666666666666666666666666666667hy_4 \end{aligned} \quad (9)$$

For N = 6, the rule is obtained by fitting a third-degree polynomial to five equally spaced discrete points. The upper limit of integration is x_6, then:

$$I = \frac{5h}{24}(11\,y_2 + y_3 + y_4 + 11\,y_5) \quad \text{or} \quad \text{Error} \approx O\left(h^4\right)$$
$$\begin{aligned} I = {} & 2.291666666666666666666666666666666666667hy_2 \\ & + 0.2083333333333333333333333333333333333333hy_3 \\ & + 0.2083333333333333333333333333333333333333hy_4 \\ & + 2.291666666666666666666666666666666666667hy_5 \end{aligned} \quad (11)$$

Generally, have the open formulas, where N is number of points, ci are integer coefficients and cr are real coefficients:

$$\begin{aligned} I = {} & \frac{num}{den}h(ci_2y_2 + ci_3y_3 + ci_4\,y_4 + \ldots + ci_{N-2}y_{N-2} + ci_{N-1}y_{N-1}) \\ \text{or} \quad I = {} & h\left(cr_2\,y_2 + cr_3\,y_3 + \ldots + cr_{N-2}\,y_{N-2} + cr_{N-1}\,y_{N-1}\right) \end{aligned} \quad (12)$$

```
restart:
for N from 2 by 1 to 105 do
assign (x, array (1..N)): assign (y, array (1..N)):
for j from 1 by 1 to N do x[j]:= 0+(j-1)*1: od:
p: = sort (factor (int (interp(x,y,xx), xx = x[1]..x [N])));
p1:= sort (simplify (p*h));
p2:=evalf (sort (expand (p*h)), 40);
for j from 1 by 1 to N do print (coeff (p2, y[j])); od:
od;
```

Fig. 1: Verification Newton-cotes closed integration formulas with maple 12.0®. (number of points N = 2..105)

Table 1: Newton-cotes closed integration formulas with points = 2.14

Rule	Integer coefficient	Real coefficient
Trapezoidal rule	num = 1	
Points (N) = 2	den = 2	
Segments (N-1) = 1	$ci_1 = ci_2 = 1$	$cr_1 = cr_2 = +0.5000000000000000000000000000000000000000$
Error»O (h^2)		
Simpson's 1/3 rule	num = 1	
Points (N) = 3	den = 3	
Segments (N-1) = 2	$ci_1 = ci_3 = 1$	$cr_1 = cr_3 = +0.3333333333333333333333333333333333333333$
Error»O (h^4)	$ci_2 = 4$	$cr_2 = +1.33333333333333333333333333333333333333$
Simpson's 3/8 rule	num = 3	
Points (N) = 4	den = 8	
Segments (N-1) = 3	$ci_1 = ci_4 = 1$	$cr_1 = cr_4 = +0.3750000000000000000000000000000000000000$
Error»O (h^4)	$ci_2 = ci_3 = 3$	$cr_2 = cr_3 = +1.1250000000000000000000000000000000000000$
Boole's rule	num = 2	
Points (N) = 5	den = 45	
Segments (N-1) = 4	$ci_1 = ci_5 = 7$	$cr_1 = cr_5 = +0.3111111111111111111111111111111111111111$
Error»O (h^6)	$ci_2 = ci_4 = 32$	$cr_2 = cr_4 = +1.4222222222222222222222222222222222222222$
	$ci_3 = 12$	$cr_3 = +0.53333333333333333333333333333333333333$
	num = 5	
Points (N) = 6	den = 288	
Segments (N-1) = 5	$ci_1 = ci_6 = 19$	$cr_1 = cr_6 = +0.3298611111111111111111111111111111111111$
Error»O (h^6)	$ci_2 = ci_5 = 75$	$cr_2 = cr_5 = +1.3020833333333333333333333333333333333333$
	$ci_3 = ci_4 = 50$	$cr_3 = cr_4 = +0.8680555555555555555555555555555555555556$
	num = 1	
Points (N) = 7	den = 140	
Segments (N-1) = 6	$ci_1 = ci_7 = 41$	$cr_1 = cr_7 = +0.2928571428571428571428571428571428571429$
Error»O (h^8)	$ci_2 = ci_6 = 216$	$cr_2 = cr_6 = +1.5428571428571428571428571428571428571429$
	$ci_3 = ci_5 = 27$	$cr_3 = cr_5 = +0.1928571428571428571428571428571428571429$
	$ci_4 = 272$	$cr_4 = +1.9428571428571428571428571428571428571429$
	num = 7	
Points (N) = 8	den = 17280	
Segments (N-1) = 7	$ci_1 = ci_8 = 751$	$cr_1 = cr_8 = +0.3042245370370370370370370370370370370370$
Error»O (h^8)	$ci_2 = ci_7 = 3577$	$cr_2 = cr_7 = +1.4490162037037037037037037037037037037037$
	$ci_3 = ci_6 = 1323$	$cr_3 = cr_6 = +0.5359375000000000000000000000000000000000$
	$ci_4 = ci_5 = 2989$	$cr_4 = cr_5 = +1.2108217592592592592592592592592592592593$
	num = 4	
Points (N) = 9	den = 14175	
Segments (N-1) = 8	$ci_1 = ci_9 = 989$	$cr_1 = cr_9 = +0.2790828924162257495590828924162257495591$
Error»O (h^{10})	$ci_2 = ci_8 = 5888$	$cr_2 = cr_8 = +1.6615167548500881834215167548500881834215$
	$ci_3 = ci_7 = -928$	$cr_3 = cr_7 = -0.2618694885361552028218694885361552028219$
	$ci_4 = ci_6 = 10496$	$cr_4 = cr_6 = +2.9618342151675485008818342151675485008818$
	$ci_5 = -4540$	$cr_5 = -1.28112874779541446208112874779541446208 11$
	num = 9	
Points (N) = 10	den = 89600	
Segments (N-1) = 9	$ci_1 = ci_{10} = 2857$	$cr_1 = cr_{10} = +0.2869754464285714285714285714285714285714$
Error»O (h^{10})	$ci_2 = ci_9 = 15741$	$cr_2 = cr_9 = +1.5811272321428571428571428571428571428571$
	$ci_3 = ci_8 = 1080$	$cr_3 = cr_8 = +0.1084821428571428571428571428571428571429$
	$ci_4 = ci_7 = 19344$	$cr_4 = cr_7 = +1.9430357142857142857142857142857142857143$
	$ci_5 = ci_6 = 5778$	$cr_5 = cr_6 = +0.5803794642857142857142857142857142857143$
	num = 5	
Points (N) = 11	den = 299376	
Segments (N-1) = 10	$ci_1 = ci_{11} = 16067$	$cr_1 = cr_{11} = +0.2683414836192613970391748169525947303725$
Error»O (h^{12})	$ci_2 = ci_{10} = 106300$	$cr_2 = cr_{10} = +1.7753594142483031371920260809149698038 59$
	$ci_3 = ci_9 = -48525$	$cr_3 = cr_9 = -0.8104357062690396023729357062690396023729$
	$ci_4 = ci_8 = 272400$	$cr_4 = cr_8 = +4.5494628827962161295494628827962161295 49$
	$ci_5 = ci_7 = -260550$	$cr_5 = cr_7 = -4.3515512265512265512265512265512265512 27$
	$ci_6 = 427368$	$cr_6 = +7.1376463043129709796376463043129709796 38$
	num = 11	
Points (N) = 12	den = 87091200	
Segments (N-1) = 11	$ci_1 = ci_{12} = 2171465$	$cr_1 = cr_{12} = +0.2742655400315990593768371546149323927102$
Error»O (h^{12})	$ci_2 = ci_{11} = 13486539$	$cr_2 = cr_{11} = +1.7034089437279541446208112874779541446 21$
	$ci_3 = ci_{10} = -3237113$	$cr_3 = cr_{10} = -0.4088615497317754262198706643151087595532$
	$ci_4 = ci_9 = 25226685$	$cr_4 = cr_9 = +3.1862408027447089947089947089947089947 09$
	$ci_5 = ci_8 = -9595542$	$cr_5 = cr_8 = -1.2119589809303350970017636684303350970 02$
	$ci_6 = ci_7 = 15493566$	$cr_6 = cr_7 = +1.9569052441578483245149911816578483245 15$

Table 1: Continued

	num = 1	
Points (N) = 13	den = 5255250	
Segments (N-1) = 12	$ci_1 = ci_{13}$ = 1364651	$cr_1 = cr_{13}$ = +0.259673849959564245278530992816707 1024214
Error»O (h^{14})	$ci_2 = ci_{12}$ = 9903168	$cr_2 = cr_{12}$ = +1.884433281004709576138147566718995290424
	$ci_3 = ci_{11}$ = -7587864	$cr_3 = cr_{11}$ = -1.443863565006422149279292136434993577851
	$ci_4 = ci_{10}$ = 35725120	$cr_4 = cr_{10}$ = +6.797986775129632272489415346558203701061
	$ci_5 = ci_9$ = -51491295	$cr_5 = cr_9$ = -9.798067646639075210503781932353360924789
	$ci_6 = ci_8$ = 87516288	$cr_6 = cr_8$ = +16.653116026830312544598258883973169687 46
	ci_7 = -87797136	cr_7 = -16.706557442557442557442557442557442557 44
	num = 13	
Points (N) = 14	den = 402361344000	
Segments (N-1) = 13	$ci_1 = ci_{14}$ = 8181904909	$cr_1 = cr_{14}$ = +0.264351348366606509794340482171170001857 8
Error»O (h^{14})	$ci_2 = ci_{13}$ = 56280729661	$cr_2 = cr_{13}$ = +1.818389108455209852365937022021678106334
	$ci_3 = ci_{12}$ = -31268252574	$cr_3 = cr_{12}$ = -1.010254313749384433908243432052955862480
	$ci_4 = ci_{11}$ = 156074417954	$cr_4 = cr_{11}$ = +5.042650005170476813995332513851032369551
	$ci_5 = ci_{10}$ = -151659573325	$cr_5 = cr_{10}$ = -4.900009612317529190875751722312568873415
	$ci_6 = ci_9$ = 206683437987	$cr_6 = cr_9$ = +6.677790334230019869900822281774662727044
	$ci_7 = ci_8$ = -43111992612	$cr_7 = cr_8$ = -1.392916870155399421272437145453018468891

Table 2: Newton-cotes closed integration formulas with points = 15.21

Rule	Integer coefficient	Real coefficient
	num = 7	
Points (N) = 15	den = 2501928000	
Segments (N-1) = 14	$ci_1 = ci_{15}$ = 90241897	$cr_1 = cr_{15}$ = +0.252482597021177268090848337761918008831 6
Error»O (h^{16})	$ci_2 = ci_{14}$ = 710986864	$cr_2 = cr_{14}$ = +1.989229125698261500730636533105668908138
	$ci_3 = ci_{13}$ = -770720657	$cr_3 = cr_{13}$ = -2.156354858732945152698239117992204411957
	$ci_4 = ci_{12}$ = 3501442784	$cr_4 = cr_{12}$ = +9.796484746163758509437521783200795546475
	$ci_5 = ci_{11}$ = -6625093363	$cr_5 = cr_{11}$ = -18.535966479051355594565471108680985224 20
	$ci_6 = ci_{10}$ = 12630121616	$cr_6 = cr_{10}$ = +35.337088562100907779920125599137944816 96
	$ci_7 = ci_9$ = -16802270373	$cr_7 = cr_9$ = -47.010102853079704931556783408635260487 11
	ci_8 = 19534438464	cr_8 = +54.654278319759801241282722764204245685 73
	num = 5	
Points (N) = 16	den = 688816128	
Segments (N-1) = 15	$ci_1 = ci_{16}$ = 35310023	$cr_1 = cr_{16}$ = +0.256309496574389152514152514152514152514 2
Error»O (h^{16})	$ci_2 = ci_{15}$ = 265553865	$cr_2 = cr_{15}$ = +1.927610680161077761524190095618667047238
	$ci_3 = ci_{14}$ = -232936065	$cr_3 = cr_{14}$ = -1.690843575892578403739118024832310546596
	$ci_4 = ci_{13}$ = 1047777585	$cr_4 = cr_{13}$ = +7.605640623153353343085485942628799771657
	$ci_5 = ci_{12}$ = -1562840685	$cr_5 = cr_{12}$ = -11.344396722662103521478521478521478521 48
	$ci_6 = ci_{11}$ = 2461884669	$cr_6 = cr_{11}$ = +17.870405242601985068502925645782788639 93
	$ci_7 = ci_{10}$ = -2000332805	$cr_7 = cr_{10}$ = -14.520078172443720713809999524285238570 95
	$ci_8 = ci_9$ = 1018807605	$cr_8 = cr_9$ = +7.395352428507597313400884829456258027687
	num = 8	
Points (N) = 17	den = 488462349375	
Segments (N-1) = 16	$ci_1 = ci_{17}$ = 15043611773	$cr_1 = cr_{17}$ = +0.246383153866392099957130907782773467318 9
Error»O (h^{18})	$ci_2 = ci_{16}$ = 127626606592	$cr_2 = cr_{16}$ = +2.090259063042242486859839973925114154045
	$ci_3 = ci_{15}$ = -179731134720	$cr_3 = cr_{15}$ = -2.943623146389408449788157226688194630927
	$ci_4 = ci_{14}$ = 832211855360	$cr_4 = cr_{14}$ = +13.629903822472069523976706602843477346 36
	$ci_5 = ci_{13}$ = -1929498607520	$cr_5 = cr_{13}$ = -31.601184574227144341609880912021521351 67
	$ci_6 = ci_{12}$ = 4177588893696	$cr_6 = cr_{12}$ = +68.420239947522362710660251898969608275 55
	$ci_7 = ci_{11}$ = -6806534407936	$cr_7 = cr_{11}$ = -111.47691389758304030185622328652380179 16
	$ci_8 = ci_{10}$ = 9368875018240	$cr_8 = cr_{10}$ = +153.44273768862986278347484558363971857 76
	ci_9 = -10234238972220	cr_9 = -167.61560411466667302334902708385434809 34
	num = 17	
Points (N) = 18	den = 3766102179840000	
Segments (N-1) = 17	$ci_1 = ci_{18}$ = 55294720874657	$cr_1 = cr_{18}$ = +0.249597650297715667408587014701065259560 3
Error»O (h^{18})	$ci_2 = ci_{17}$ = 450185515446285	$cr_2 = cr_{17}$ = +2.032115273864391922530976763780996585229
	$ci_3 = ci_{16}$ = -542023437008852	$cr_3 = cr_{16}$ = -2.446667134650592709501072228879401131018
	$ci_4 = ci_{15}$ = 2428636525764260	$cr_4 = cr_{15}$ = +10.962745822192869799286980356939204675 83
	$ci_5 = ci_{14}$ = -4768916800123440	$cr_5 = cr_{14}$ = -21.526655871440733171883965534759185552 84
	$ci_6 = ci_{13}$ = 8855416648684984	$cr_6 = cr_{13}$ = +39.972915188944872024218732037662078814 34
	$ci_7 = ci_{12}$ = -10905371859796660	$cr_7 = cr_{12}$ = -49.226312182644877136398667850754858397 42
	$ci_8 = ci_{11}$ = 10069615750132836	$cr_8 = cr_{11}$ = +45.453750211188059702031262877823724384 57
	$ci_9 = ci_{10}$ = -3759785974054070	$cr_9 = cr_{10}$ = -16.971488957751706097692833436513624638 26
	num = 3	
Points (N) = 19	den = 2534852320000	

Table 2: Continued

Segments (N-1) = 18	$ci_1 = ci_{19} = 203732352169$	$cr_1 = cr_{19} = +0.2411174219833840260958476665812231617501$
Error»O (h^{20})	$ci_2 = ci_{18} = 1848730221900$	$cr_2 = cr_{18} = +2.1879738799536850336117411368564461380 54$
	$ci_3 = ci_{17} = -3212744374395$	$cr_3 = cr_{17} = -3.8022858559211843946790557013593596647 87$
	$ci_4 = ci_{16} = 15529830312096$	$cr_4 = cr_{16} = +18.379568138426304850769373420539149988 82$
	$ci_5 = ci_{15} = -42368630685840$	$cr_5 = cr_{15} = -50.143312513574755313556097027380277522 44$
	$ci_6 = ci_{14} = 103680563465808$	$cr_6 = cr_{14} = +122.70604008892478596149538210573150865 06$
	$ci_7 = ci_{13} = -198648429867720$	$cr_7 = cr_{13} = -235.10059536847495715253344620881109160 63$
	$ci_8 = ci_{12} = 319035784479840$	$cr_8 = cr_{12} = +377.57913780141637600410583287944758848 91$
	$ci_9 = ci_{11} = -419127951114198$	$cr_9 = cr_{11} = -496.03830701371746974198481117038013480 80$
	$ci_{10} = 461327344340680$	$cr_{10} = +545.9813268419676614533504657975498943465$
	num = 19	
Points (N) = 20	den = 5377993912811520000	
Segments (N-1) = 19	$ci_1 = ci_{20} = 69028763155644023$	$cr_1 = cr_{20} = +0.2438728122828207419466199855927763551718$
Error»O (h^{20})	$ci_2 = ci_{19} = 603652082270808125$	$cr_2 = cr_{19} = +2.1326520165489292228358723655401997520 74$
	$ci_3 = ci_{18} = -926840515700222955$	$cr_3 = cr_{18} = -3.2744495594079346437440282617242290000 28$
	$ci_4 = ci_{17} = 4301581538450500095$	$cr_4 = cr_{17} = +15.197125648629170556529342443399424216 44$
	$ci_5 = ci_{16} = -10343692234243192788$	$cr_5 = cr_{16} = -36.543394365405329892940620101854966171 06$
	$ci_6 = ci_{15} = 22336420328479961316$	$cr_6 = cr_{15} = +78.912693677493332913769317497475427314 03$
	$ci_7 = ci_{14} = -35331888421114781580$	$cr_7 = cr_{14} = -124.82458903532563157093669530125594032 64$
	$ci_8 = ci_{11} = 43920768370565135580$	$cr_8 = cr_{13} = +155.16837924505544380673010442819601050 79$
	$ci_9 = ci_{12} = -37088370261379851390$	$cr_9 = cr_{12} = -131.03009159001138647679338608488046040 84$
	$ci_{10} = ci_{11} = 15148337305921759574$	$cr_{10} = cr_{11} = +53.517801150140585342603473029511757760 26$
	num = 1	
Points (N) = 21	den = 82324272054024	
Segments (N-1) = 20	$ci_1 = ci_{21} = 19470140241329$	$cr_1 = cr_{21} = +0.2365054649806320638934570035459272429452$
Error»O (h^{22})	$ci_2 = ci_{20} = 187926090380000$	$cr_2 = cr_{20} = +2.2827543528921394997499043254887351900 28$
	$ci_3 = ci_{19} = -389358194177500$	$cr_3 = cr_{19} = -4.7295674102285392846208636219418067090 62$
	$ci_4 = ci_{18} = 1985969159340000$	$cr_4 = cr_{18} = +24.123737869637513288070169416442542819 52$
	$ci_5 = ci_{17} = -6208948835889375$	$cr_5 = cr_{17} = -75.420634534306609354755299095978640681 32$
	$ci_6 = ci_{16} = 17019387776517504$	$cr_6 = cr_{16} = +206.73596439879602287062366807734937462 66$
	$ci_7 = ci_{15} = -37389734671290000$	$cr_7 = cr_{15} = -454.17631687959024595925994404538361156 59$
	$ci_8 = ci_{14} = 68869287574320000$	$cr_8 = cr_{14} = +836.56114844387109206952122360749434159 38$
	$ci_9 = ci_{13} = -105499014813701250$	$cr_9 = cr_{13} = -1281.5055898030800930311099629777147055 74$
	$ci_{10} = ci_{12} = 136324521798440000$	$cr_{10} = cr_{12} = +1655.9456694494570344170493627037883217 39$
	$ci_{11} = -148192526607280936$	$cr_{11} = -1800.1073427048578931583234307861809573 63$

Table 3: Newton-cotes open integration formulas with points = 3.16

Rule	Integer coefficient	Real coefficient	
Midpoint rule	num = 2		
Points (N) = 3	den = 1		
Segments (N-1) = 2	$ci_2 = 1$		$cr_2 = +2.0000000000000000000000000000000000000000$
Error»O (h^2)			
Points (N) = 4	num = 3		
Segments (N-1) = 3	den = 2		
Error»O (h^2)	$ci_2 = ci_3 = 1$		$cr_2 = cr_3 = +1.5000000000000000000000000000000000000000$
Points (N) = 5	num = 4	den = 3	
Segments (N-1) = 4	$ci_2 = ci_4 = 2$		$cr_2 = cr_4 = +2.6666666666666666666666666666666666666667$
Error»O (h^4)	$ci_3 = -1$		$cr_3 = -1.3333333333333333333333333333333333333333$
Points (N) = 6	num = 5	den = 24	
Segments (N-1) = 5	$ci_2 = ci_5 = 11$		$cr_2 = cr_5 = +2.2916666666666666666666666666666666666667$
Error»O (h^4)	$ci_3 = ci_4 = 1$		$cr_3 = cr_4 = +0.2083333333333333333333333333333333333333$
Points (N) = 7	num = 3	den = 10	
Segments (N-1) = 6	$ci_2 = ci_6 = 11$		$cr_2 = cr_6 = +3.3000000000000000000000000000000000000000$
Error»O (h^6)	$ci_2 = ci_5 = -14$		$cr_3 = cr_5 = -4.2000000000000000000000000000000000000000$
	$ci_4 = 26$		$cr_4 = +7.8000000000000000000000000000000000000000$
Points (N) = 8	num = 7	den = 1440	
Segments (N-1) = 7	$ci_2 = ci_7 = 611$		$cr_2 = cr_7 = +2.9701388888888888888888888888888888888889$
Error»O (h^6)	$ci_3 = ci_6 = -453$		$cr_3 = cr_6 = -2.2020833333333333333333333333333333333333$
	$ci_4 = ci_6 = 562$		$cr_4 = cr_5 = +2.7319444444444444444444444444444444444444$
Points (N) = 9	num = 8	den = 945	
Segments (N-1) = 8	$ci_2 = ci_8 = 460$		$cr_2 = cr_8 = +3.8941798941798941798941798941798941798 94$
Error»O (h^8)	$ci_3 = ci_7 = -954$		$cr_3 = cr_7 = -8.0761904761904761904761904761904761904 76$
	$ci_4 = ci_6 = 2196$		$cr_4 = cr_6 = +18.590476190476190476190476190476190476 19$
	$ci_5 = -2459$		$cr_5 = -20.816931216931216931216931216931216931 22$

Table 3: Continued

Points (N) = 10	num = 9	den = 4480	
Segments (N-1) = 9	$ci_2 = ci_9 = 1787$		$cr_2 = cr_9 = +3.5899553571428571428571428571428571428571428 57$
Error»O (h^8)	$ci_3 = ci_8 = -2803$		$cr_3 = cr_8 = -5.6310267857142857142857142857142857142857142 86$
	$ci_4 = ci_7 = 4967$		$cr_4 = cr_7 = +9.9783482142857142857142857142857142857142857 14$
	$ci_5 = ci_6 = -1711$		$cr_5 = cr_6 = -3.4372767857142857142857142857142857142857142 86$
	num = 5		
Points (N) = 11	den = 4536		
Segments (N-1) = 10	$ci_2 = ci_{10} = 4045$		$cr_2 = cr_{10} = +4.4587742504409171075837742504409171075 84$
Error»O (h^{10})	$ci_3 = ci_9 = -11690$		$cr_3 = cr_9 = -12.885802469135802469135802469135802469 14$
	$ci_4 = ci_8 = 33340$		$cr_4 = cr_8 = +36.750440917107583774250440917107583774 25$
	$ci_5 = ci_7 = -55070$		$cr_5 = cr_7 = -60.703262786596119929453262786596119929 45$
	$ci_6 = 67822$		$cr_6 = +74.759700176366843033509700176366843033 51$
	num = 11		
Points (N) = 12	den = 7257600		
Segments (N-1) = 11	$ci_2 = ci_{11} = 2752477$		$cr_2 = cr_{11} = +4.1717988040123456790123456790123456790 12$
Error»O (h^{10})	$ci_3 = ci_{10} = -6603199$		$cr_3 = cr_{10} = -10.008155450837742504409171075837742504 41$
	$ci_4 = ci_9 = 15673880$		$cr_4 = cr_9 = +23.756156305114638447971781305114638447 97$
	$ci_5 = ci_8 = -17085616$		$cr_5 = cr_8 = -25.895857583774250440917107583774250440 92$
	$ci_6 = ci_7 = 8891258$		$cr_6 = cr_7 = +13.476057925485008818342151675485008818 34$
	num = 1		
Points (N) = 13	den = 1925		
Segments (N-1) = 12	$ci_2 = ci_{12} = 9626$		$cr_2 = cr_{12} = +5.0005194805194805194805194805194805194 81$
Error»O (h^{12})	$ci_3 = ci_{11} = -35771$		$cr_3 = cr_{11} = -18.582337662337662337662337662337662337 66$
	$ci_4 = ci_{10} = 123058$		$cr_4 = cr_{10} = +63.926233766233766233766233766233766233 77$
	$ci_5 = ci_9 = -266298$		$cr_5 = cr_9 = -138.33662337662337662337662337662337662 34$
	$ci_6 = ci_8 = 427956$		$cr_6 = cr_8 = +222.31480519480519480519480519480519480 52$
	$ci_7 = -494042$		$cr_7 = -256.64519480519480519480519480519480519 48$
	num = 13		
Points (N) = 14	den = 958003200		
Segments (N-1) = 13	$ci_2 = ci_{13} = 348289723$		$cr_2 = cr_{13} = +4.7262539404878814601036823259045481267 70$
Error»O (h^{12})	$ci_3 = ci_{12} = -1126407423$		$cr_3 = cr_{12} = -15.285227125546135962802629469296135962 80$
	$ci_4 = ci_{11} = 3371637557$		$cr_4 = cr_{11} = +45.752757653627879322323766768211212655 66$
	$ci_5 = ci_{10} = -5718293865$		$cr_5 = cr_{10} = -77.596630413134319384319384319384319384 32$
	$ci_6 = ci_9 = 6277879038$		$cr_6 = cr_9 = +85.190140799112153278819945486612153278 82$
	$ci_7 = ci_8 = -2674103430$		$cr_7 = cr_8 = -36.287294854547458714125380792047458714 13$
	num = 7		
Points (N) = 15	den = 416988000		
Segments (N-1) = 14	$ci_2 = ci_{14} = 329062237$		$cr_2 = cr_{14} = +5.5239854839947432540025132617725210317 80$
Error»O (h^{14})	$ci_3 = ci_{13} = -1497122214$		$cr_3 = cr_{13} = -25.132271187660076548965437854326743215 63$
	$ci_4 = ci_{12} = 6058248882$		$cr_4 = cr_{12} = +101.70015006187228409450631672853895076 12$
	$ci_5 = ci_{11} = -16159538710$		$cr_5 = cr_{11} = -271.27104609724980095350465720836091206 46$
	$ci_6 = ci_{10} = 32215733235$		$cr_6 = cr_{10} = +540.80724779849779849779849779849779849 78$
	$ci_7 = ci_9 = -47966447844$		$cr_7 = cr_9 = -805.21534170767504100837434170767504100 84$
	$ci_8 = 54874104828$		$cr_8 = +921.17455129644018532907421796310685199 57$
	num = 5		
Points (N) = 16	den = 172204032		
Segments (N-1) = 15	$ci_2 = ci_{15} = 181146041$		$cr_2 = cr_{15} = +5.2596341356281367442081727796013510299 22$
Error»O (h^{14})	$ci_3 = ci_{14} = -737951959$		$cr_3 = cr_{14} = -21.426674812120543147328861614575900290 19$
	$ci_4 = ci_{13} = 2671853466$		$cr_4 = cr_{13} = +77.578133187961591979449122306265163408 02$
	$ci_5 = ci_{12} = -6013831334$		$cr_5 = cr_{12} = -174.61354604054799367299367299367299367 30$
	$ci_6 = ci_{11} = 9451804423$		$cr_6 = cr_{11} = +274.43621131356552673516959231244945530 66$
	$ci_7 = ci_{10} = -9336416457$		$cr_7 = cr_{10} = -271.08588424340726238047666619095190523 76$
	$ci_8 = ci_9 = 4041701868$		$cr_8 = cr_9 = +117.35212645892054374197231340088482945 63$

The Table 3 and 4 shows of coefficients for higher order open integration formulas. Figure 2 shows the generation of rules in Maple 12.0®.

Newton-cotes semi-closed or semi-open integration equation: In the semi-open integration formulas, the last (y_N) point does not appear in equation. In the semi-closed integration formulas, the first (y_1) point does not appear in equation (Zhang *et al*., 2009). The rule for a double interval is obtained by fitting a zero-degree polynomial to three discrete points. When N is odd, the semi-closed or semi-open integration formulas are same as open rules. The upper limit of integration is $x_3 = x_1+2h$, then the Integral (I) have the formula:

$$I = 2h\left(y_2\right) \quad \text{or} \quad I = 2hy_2 \qquad \text{Error} \approx O\left(h^2\right) \tag{13}$$

```
restart:
for N from 3 by 1 to 105 do
assign (x, array (1..N-2)): assign(y, array (1..N-2)):
for j from 1 by 1 to N-2 do x[j]:=0+j*1: od:
p:=sort(factor (int ( interp(x,y,xx), xx=0..x[N-2]+1) )):
p1:= sort (simplify (p*h));
p2:=evalf (sort (expand (p*h)), 40):
for j from 1 by 1 to N-2 do print (coeff (p2, y[j])); od:
od;
```

Fig. 2: Verification Newton-cotes open integration formulas with maple 12.0®. with number of points N = 3..105

For three intervals, the semi-open rule is obtained by fitting a first-degree polynomial to four discrete points. The upper limit of integration is $x_4 = x_1+3h$, then the Integral (I) have the formula:

$$I=\frac{3h}{4}(y_1+3y_3) \quad \text{or} \quad I=0.75hy_1+2.25hy_3 \quad \text{Error}\approx O(h^3) \qquad (14)$$

For N = 5, the rule is obtained by fitting a second-degree polynomial to four equally spaced discrete points, same of the open formula. The upper limit of integration is x_5, then:

$$I=\frac{4h}{3}(2y_2-y_3+2y_4) \quad \text{or} \quad \text{Error}\approx O(h^4)$$

$$I=2.666666666666666666666666666666666666667hy_2 -1.333333333333333333333333333333333333333hy_3 +2.666666666666666666666666666666666666667hy_4 \qquad (15)$$

For N = 6, the rule is obtained by fitting a third-degree polynomial to five equally spaced discrete points. The upper limit of integration is x_6, then:

$$I=\frac{5h}{144}(19y_1\text{-}10\,y_2+120\,y_3\text{-}70\,y_4+85\,y_5) \text{ or Error}\approx O(h^5)$$

$$I=0.6597222222222222222222222222222222222222hy_1 -0.3472222222222222222222222222222222222222hy_2 +4.166666666666666666666666666666666666667hy_3 -2.430555555555555555555555555555555555556hy_4 +2.951388888888888888888888888888888888889hy_5 \qquad (16)$$

Generally, have the semi-open formulas, where N is number of points:

$$I=\frac{num}{den}h(ci_1y_1+ci_2y_2+ci_3y_3+ci_4\,y_4+\ldots+ci_{N-2}y_{N-2}+ci_{N-1}y_{N-1}) \qquad (17)$$

$$\text{or} \quad I=h\begin{pmatrix}cr_1\,y_1+cr_2\,y_2+cr_3\,y_3+\ldots\\+cr_{N-2}\,y_{N-2}+cr_{N-1}\,y_{N-1}\end{pmatrix}$$

```
restart:
for N from 3 by 1 to 105 do
assign (x, array (1..N-1)): assign(y, array (1..N-1)):
for j from 1 by 1 to N-1 do x[j]:=0+(j-1)*1: od:
p:=sort(factor(int (interp (x,y,xx), xx = 0..x[N-1]+1) )):
p1:= sort (simplify (p*h));
p2:= evalf (sort (expand (p*h)), 40):
for j from 2 by 1 to N-1 do print (coeff (p2, y[j])); od:
od;
```

Fig. 3: Verification Newton-cotes semi-open integration formulas with maple 12.0®. with N = 3..105

The Table 5 and 6 shows of coefficients for higher order open integration formulas. Figure 3 shows the generation of rules in Maple 12.0®.

The semi-closed integration formulas are same as semi-open rules. For examples, N = 3:

$$I=2h(y_2) \quad \text{or} \quad I=2hy_2 \qquad \text{Error}\approx O(h^2) \qquad (18)$$

For N = 4:

$$I=\frac{3h}{4}(y_2+3y_4) \quad \text{or} \quad I=0.75hy_2+2.25hy_4 \quad \text{Error}\approx O(h^3) \qquad (19)$$

Generally, have the semi-closed formulas, where N is number of points, ci are integer coefficients and cr are real coefficients:

$$I=\frac{num}{den}h(ci_2y_2+ci_3y_3+ci_4\,y_4+\ldots+ci_{N-2}y_{N-2}+ci_{N-1}y_{N-1}+ci_Ny_N)$$

$$\text{or} \quad I=h\begin{pmatrix}cr_2\,y_2+cr_3\,y_3+\ldots\\+cr_{N-2}\,y_{N-2}+cr_{N-1}\,y_{N-1}+cr_N\,y_N\end{pmatrix} \qquad (20)$$

The semi-open or semi-closed rules can be used on type of improper integral-that is, one with a lower limit of -∞ or an upper limit of +∞. Such integrals usually can be evaluated by making a change of variable that transforms the infinite range to one that is finite (Choi *et al*., 2003). The following identity serves this purpose and works for any function that decreases toward zero at least as fast $1/x^2$ as x approaches infinity:

$$\int_a^b f(x)dx=\int_{1/b}^{1/a}\frac{1}{w^2}f\left(\frac{1}{w}\right)dw \qquad (21)$$

Table 4: Newton-cotes open integration formulas with points = 17.23

Rule	Integer coefficient	Real coefficient
	num = 16	
Points (N) = 17	den = 1915538625	
Segments (N-1) = 16	$ci_2 = ci_{16}$ = 722204696	$cr_2 = cr_{16}$ = +6.0323895249045160861739344984494896311147
Error»O (h^{16})	$ci_3 = ci_{15}$ = -3892087348	$cr_3 = cr_{15}$ = -32.509601610356460444643866160621010709 19
	$ci_4 = ci_{14}$ = 18150263624	$cr_4 = cr_{14}$ = +151.604469987651645499970014961196619044 9
	$ci_5 = ci_{13}$ = -57468376538	$cr_5 = cr_{13}$ = -480.018524611060766263588133076669231872 1
	$ci_6 = ci_{12}$ = 137035461016	$cr_6 = cr_{12}$ = +1144.621856035923055323408057094124113524
	$ci_7 = ci_{11}$ = -249560348012	$cr_7 = cr_{11}$ = -2084.513210059650976758560532810973728081
	$ci_8 = ci_{10}$ = 355819203336	$cr_8 = cr_{10}$ = +2972.066017920155486293052430618568184706
	ci_9 = -399697102923	cr_9 = -3338.566794375132999471623810248148872487
	num = 17	
Points (N) = 18	den = 62768369664000	
Segments (N-1) = 17	$ci_2 = ci_{17}$ = 35310023	$cr_2 = cr_{17}$ = +5.7760800283301269336597819842969754786 33
Error»O (h^{16})	$ci_3 = ci_{16}$ = 265553865	$cr_3 = cr_{16}$ = -28.404822765613022119994121757790188125 29
	$ci_4 = ci_{15}$ = -232936065	$cr_4 = cr_{15}$ = +120.785711953187763459065266825407918882 3
	$ci_5 = ci_{14}$ = 1047777585	$cr_5 = cr_{14}$ = -336.019695246562474106703604058101412598 8
	$ci_6 = ci_{13}$ = -1562840685	$cr_6 = cr_{13}$ = +675.947728147524392581298445495976360173 9
	$ci_7 = ci_{12}$ = 2461884669	$cr_7 = cr_{12}$ = -957.761759266329906503655401362632403196 8
	$ci_8 = ci_{11}$ = -2000332805	$cr_8 = cr_{11}$ = +902.072886032948230248301897331879695195 4
	$ci_9 = ci_{10}$ = 1018807605	$cr_9 = cr_{10}$ = -373.896128883485110491972264459036945809 4
	num = 9	
Points (N) = 19	den = 9529520000	
Segments (N-1) = 18	$ci_2 = ci_{18}$ = 6912171129	$cr_2 = cr_{18}$ = +6.5280874756545975033369991353184630495 55
Error»O (h^{18})	$ci_3 = ci_{17}$ = -43087461474	$cr_3 = cr_{17}$ = -40.693251419378940387343748688286503412 55
	$ci_4 = ci_{16}$ = 227788759000	$cr_4 = cr_{16}$ = +215.131384476867670144981069350817249976 9
	$ci_5 = ci_{15}$ = -834322842510	$cr_5 = cr_{15}$ = -787.962623782729875166849956765923152477 8
	$ci_6 = ci_{14}$ = 2317367615100	$cr_6 = cr_{14}$ = +2188.600111642559121550718189373651558526
	$ci_7 = ci_{13}$ = -4988390746282	$cr_7 = cr_{13}$ = -4711.204417068016017595849528622637866335
	$ci_8 = ci_{12}$ = 8524579147752	$cr_8 = cr_{12}$ = +8050.899975000629622478361974160293488025
	$ci_9 = ci_{11}$ = -11696802277350	$cr_9 = cr_{11}$ = -11046.854458162635683644087005431543246 67
	ci_{10} = 12990970309270	cr_{10} = +12269.110383674099010233464014976620018 64
	num = 19	
Points (N) = 20	den = 64023737057280000	
Segments (N-1) = 19	$ci_2 = ci_{19}$ = 21156441141866149	$cr_2 = cr_{19}$ = +6.2784898253568818359284121206173977899 95
Error»O (h^{18})	$ci_3 = ci_{18}$ = -121972899306097215	$cr_3 = cr_{18}$ = -36.197279217588734806537726316749036948 23
	$ci_4 = ci_{17}$ = 596043470364791516	$cr_4 = cr_{17}$ = +176.884800192139322467138392891410147695 4
	$ci_5 = ci_{16}$ = -1967193294708433100	$cr_5 = cr_{16}$ = -583.793985128055074821155867772045107176 7
	$ci_6 = ci_{15}$ = 4792224378449610000	$cr_6 = cr_{15}$ = +1422.164143731269979555752198142487591601
	$ci_7 = ci_{14}$ = -8635040534820624232	$cr_7 = cr_{14}$ = -2562.577220614401768069350071286648386624
	$ci_8 = ci_{13}$ = 11419549616838153340	$cr_8 = cr_{13}$ = +3388.921870115258482018911113388410480087
	$ci_9 = ci_{12}$ = -10248543211438519308	$cr_9 = cr_{12}$ = -3041.408233373194120867756294149073483029
	$ci_{10} = ci_{11}$ = 4175787902007892850	$cr_{10} = cr_{11}$ = +1239.227414469215032687069842981590396606
	num = 5	
Points (N) = 21	den = 1247337455364	
Segments (N-1) = 20	$ci_2 = ci_{20}$ = 1749481500626	$cr_2 = cr_{20}$ = +7.0128636525047807776190443964072800489 32
Error»O (h^{20})	$ci_3 = ci_{19}$ = -12389954060697	$cr_3 = cr_{19}$ = -49.665605756548631424377694295667982868 66
	$ci_4 = ci_{18}$ = 73278572831682	$cr_4 = cr_{18}$ = +293.739967947558066126611153458799599777 1
	$ci_5 = ci_{17}$ = -304672055470086	$cr_5 = cr_{17}$ = -1221.289612365468958918554481275996132751
	$ci_6 = ci_{16}$ = 966316491145704	$cr_6 = cr_{16}$ = +3873.516693458515541474781051053405349250
	$ci_7 = ci_{15}$ = -2400158698258188	$cr_7 = cr_{15}$ = -9621.128139528889042469653401485523548124
	$ci_8 = ci_{14}$ = 4782407754794376	$cr_8 = cr_{14}$ = +19170.464793742468685090308138487774214 71
	$ci_9 = ci_{13}$ = -7751977518223986	$cr_9 = cr_{13}$ = -31074.099013413301181689888699658169499 39
	$ci_{10} = ci_{12}$ = 10322815990097148	$cr_{10} = cr_{12}$ = +41379.403567596418485962087678277728046 82
	ci_{11} = -11349750778891702	cr_{11} = -45495.911030666515489857865577917514654 96
	num = 7	
Points (N) = 22	den = 136216903680000	
Segments (N-1) = 21	$ci_2 = ci_{21}$ = 131721567613331	$cr_2 = cr_{21}$ = +6.7689908402219600356724244108145036937 61
Error»O (h^{20})	$ci_3 = ci_{20}$ = -871503959599375	$cr_3 = cr_{20}$ = -44.785394120592779869594522264800902571 80
	$ci_4 = ci_{19}$ = 4813298466509865	$cr_4 = cr_{19}$ = +247.348811750417369345977487424855845908 5
	$ci_5 = ci_{18}$ = -18345435138969285	$cr_5 = cr_{18}$ = -942.746770066540063348472670260595957190 4
	$ci_6 = ci_{17}$ = 52322284124735964	$cr_6 = cr_{17}$ = +2688.770475458451912449167189879264182670
	$ci_7 = ci_{16}$ = -113381582504747148	$cr_7 = cr_{16}$ = -5826.524139747866833908641667929593626188
	$ci_8 = ci_{15}$ = 188254898608060740	$cr_8 = cr_{15}$ = +9674.161243248905274191591432303506606912
	$ci_9 = ci_{14}$ = -234658964587522740	$cr_9 = cr_{14}$ = -12058.802598915887940406310665598591295 19

Table 4: Continued

	$ci_{10} = ci_{13} = 203086455887932170$	$cr_{10} = cr_{13} = +10436.3346457731286907489923647044390078$4
	$ci_{11} = ci_{12} = -81146847108493522$	$cr_{11} = cr_{12} = -4170.02526422023758923838137266929836589$3
	num = 11	
Points (N) = 23	den = 136073176948800000	
Segments (N-1) = 22	$ci_2 = ci_{22} = 92630057200320343$	$cr_2 = cr_{22} = +7.4881078846782191077637940232519613618$67
Error»O (h^{22})	$ci_3 = ci_{21} = -734938747634165690$	$cr_3 = cr_{21} = -59.4116078220007820533685344991428322891$0
	$ci_4 = ci_{20} = 4810326117267632170$	$cr_4 = cr_{20} = +388.861261833062444598120885818839883145$6
	$ci_5 = ci_{19} = -22377003329240798370$	$cr_5 = cr_{19} = -1808.93135694381610231326769447324145270$0
	$ci_6 = ci_{18} = 79806119136903527115$	$cr_6 = cr_{18} = +6451.43539814795601230193477315414676020$6
	$ci_7 = ci_{17} = -224650147742325069072$	$cr_7 = cr_{17} = -18160.4610149977711166388500003340784790$9
	$ci_8 = ci_{16} = 511409287186029264120$	$cr_8 = cr_{16} = +41341.7418861545291170140893439352957446$5
	$ci_9 = ci_{15} = -956236430810735088960$	$cr_9 = cr_{15} = -77301.0594356586546198279997279345773345$9
	$ci_{10} = ci_{14} = 1484914688541488498910$	$cr_{10} = cr_{14} = +120038.805150425497243382400477510559663$4
	$ci_{11} = ci_{13} = -1928478457946599022420$	$cr_{11} = cr_{13} = -155895.992972916801132917916791237977340$0
	$ci_{12} = 2103160001429187403708$	$cr_{12} = +170017.049167786641434694186948073846851$8

Table 5: Newton-cotes semi-open integration formulas with points = 4, 6, 8, 10, 12, 14, 16

Rule	Integer coefficient	Real coefficient	
	num = 3	den = 4	
Points (N) = 4	$ci_1 = 1$		$cr_1 = +0.75000000000000000000000000000000000000$00
Segments (N-1) = 3	$ci_2 = 0$		$cr_3 = +0.00000000000000000000000000000000000000$00
Error»O (h^3)	$ci_3 = 3$		$cr_3 = +2.25000000000000000000000000000000000000$00
	num = 5	den = 144	
Points (N) = 6	$ci_1 = 19$		$cr_1 = +0.65972222222222222222222222222222222222$22
Segments (N-1) = 5	$ci_2 = -10$		$cr_2 = -0.34722222222222222222222222222222222222$22
Error»O (h^5)	$ci_3 = 120$		$cr_3 = +4.16666666666666666666666666666666666666$67
	$ci_4 = -70$		$cr_4 = -2.43055555555555555555555555555555555555$56
	$ci_5 = 85$		$cr_5 = +2.95138888888888888888888888888888888888$89
	num = 7	den = 8640	
Points (N) = 8	$ci_1 = 751$		$cr_1 = +0.60844907407407407407407407407407407407$41
Segments (N-1) = 7	$ci_2 = -840$		$cr_2 = -0.68055555555555555555555555555555555555$56
Error»O (h^7)	$ci_3 = 8547$		$cr_3 = +6.92465277777777777777777777777777777777$8
	$ci_4 = -11648$		$cr_4 = -9.43703703703703703703703703703703703703$7
	$ci_5 = 14637$		$cr_5 = +11.8586805555555555555555555555555555555$6
	$ci_6 = -7224$		$cr_6 = -5.85277777777777777777777777777777777777$8
	$ci_7 = 4417$		$cr_7 = +3.57858796296296296296296296296296296296$3
	num = 9	den = 44800	
Points (N) = 10	$ci_1 = 2857$		$cr_1 = +0.57395089285714285714285714285714285714$29
Segments (N-1) = 9	$ci_2 = -4986$		$cr_2 = -1.00165178571428571428571428571428571428$6
Error»O (h^9)	$ci_3 = 51966$		$cr_3 = +10.4395982142857142857142857142857142857$1
	$ci_4 = -110322$		$cr_4 = -22.1629017857142857142857142857142857142$9
	$ci_5 = 182880$		$cr_5 = +36.7392857142857142857142857142857142857$1
	$ci_6 = -177102$		$cr_6 = -35.5785267857142857142857142857142857142$9
	$ci_7 = 129666$		$cr_7 = +26.0489732142857142857142857142857142857$1
	$ci_8 = -50886$		$cr_8 = -10.2226339285714285714285714285714285714$3
	$ci_9 = 20727$		$cr_9 = +4.16390625000000000000000000000000000000$0
	num = 11	den = 43545600	
Points (N) = 12	$ci_1 = 2171465$		$cr_1 = +0.54853108006319811875367430922986478542$03
Segments (N-1) = 11	$ci_2 = -5199788$		$cr_2 = -1.31351199661963550852439741328630217519$1
Error»O (h^{11})	$ci_3 = 58096731$		$cr_3 = +14.6757431520061728395061728395061728395$1
	$ci_4 = -166532520$		$cr_4 = -42.0675733024691358024691358024691358024$7
	$ci_5 = 353493954$		$cr_5 = +89.2956692294973544973544973544973544973$5
	$ci_6 = -493861632$		$cr_6 = -124.753774250440917107583774250440917107$6
	$ci_7 = 509355198$		$cr_7 = +128.667584738756613756613756613756613756$6
	$ci_8 = -363089496$		$cr_8 = -91.7195871913580246913580246913580246913$6
	$ci_9 = 191759205$		$cr_9 = +48.4400549079585537918871252204585537918$9
	$ci_1 = -61333844$		$cr_{10} = -15.4934662514697236919459141681363903586$1
	$ci_{11} = 18686327$		$cr_{11} = +4.72032988407554379776601998824221046443$3
	num = 13	den = 201180672000	
Points (N) = 14	$ci_1 = 8181904909$		$cr_1 = +0.52870269673321301958868096434234000371$57
Segments (N-1) = 13	$ci_2 = -25042017078$		$cr_2 = -1.61817842031067477496048924620353191781$8
Error»O (h^{13})	$ci_3 = 303460165164$		$cr_3 = +19.6091508588459233300503141772983042824$3
	$ci_4 = -1091975193010$		$cr_4 = -70.5618356276789849871860453871035881617$9
	$ci_5 = 2849201218305$		$cr_5 = +184.111204469806125312077693030073982454$9

Table 5: Continued

	ci_{6} = -5161714089948	cr_{6} = -333.542395013592558235415378272521129664
	ci_{7} = 6998518415616	cr_{7} = +452.2339969269413713858158302602747047191
	ci_{8} = -7041630408228	cr_{8} = -455.0198306672521702283607045511807416569
	ci_{9} = 5368397527935	cr_{9} = +346.8979756820525979752170228360704551181
	ci_{10} = -3000860791630	cr_{10} = -193.9112236944411836938291964746991202018
	ci_{11} = 1248049610964	cr_{11} = +80.64713563801993861517671041480565290089
	ci_{12} = -334728417738	cr_{12} = -21.62965948634469219786680104140421600739
	ci_{13} = 81322746739	cr_{13} = +5.254956637221094479692363290246888130486
	num = 5	
Points (N) = 16	den = 344408064	
Segments (N-1) = 15	ci_{1} = 35310023	cr_{1} = +0.5126189931487783050283050283050283050283
Error»O (h^{15})	ci_{2} = -132048240	cr_{2} = -1.917031768454759526188097616669045240474
	ci_{3} = 1737308175	cr_{3} = +25.22165356441828261024689596118167546739
	ci_{4} = -7509141440	cr_{4} = -109.0151803181937110508539079967651396223
	ci_{5} = 23317670355	cr_{5} = +338.5180661013790896603396603396603396603
	ci_{6} = -51787057200	cr_{6} = -751.8270129702886399314970743542172113601
	ci_{7} = 87363166155	cr_{7} = +1268.308952182373987619523333809048094762
	ci_{8} = -113100595200	cr_{8} = -1641.956258027686599115170543741972313401
	ci_{9} = 114119402805	cr_{9} = +1656.746962884701793741972313400884829456
	ci_{10} = -89363498960	cr_{10} = -1297.349108527261429047143332857618571904
	ci_{11} = 54248941869	cr_{11} = +787.5678234554926100685029256457827886399
	ci_{12} = -24880511040	cr_{12} = -361.2068595467032967032967032967032967033
	ci_{13} = 8556919025	cr_{13} = +124.2264615645004177370248798820227391656
	ci_{14} = -1970244240	cr_{14} = -28.60334071620343941772513201084629656058
	ci_{15} = 397602105	cr_{15} = +5.772253128776915049236477807906379334951

Table 6: Newton-cotes semi-open integration formulas with points = 18, 20, 22

Rule	Integer coefficient	Real coefficient
	num = 17	
Points (N) = 18	den = 1883051089920000	
Segments (N-1) = 17	ci_{1} = 55294720874657	cr_{1} +0.4991953005954313348171740294021305191205
Error»O (h^{17})	ci_{2} = -244912369711442	cr_{2} = -2.211044781196774423415002486137112827295
	ci_{3} = 3489029300972250	cr_{3} = +31.49861330583873805806676177046547416918
	ci_{4} = -17585886834501250	cr_{4} = -158.7636563802537840385521896397851718251
	ci_{5} = 63416259440780110	cr_{5} = +572.5157518371225552605531294537761322006
	ci_{6} = -166654158061846266	cr_{6} = -1504.537344853319677900117714932529747345
	ci_{7} = 336711046842479186	cr_{7} = +3039.794207901884222712274226089628793920
	ci_{8} = -532651057910098250	cr_{8} = -4808.721352778786240060168999028493443543
	ci_{9} = 670227439244428800	cr_{9} = +6050.747389779716168605057493946382835272
	ci_{10} = -673987225218482870	cr_{10} = -6084.690367695219580800443160819410084548
	ci_{11} = 542720673660231086	cr_{11} = +4899.628853201162359464231524784140892313
	ci_{12} = -347616418702275846	cr_{12} = -3138.246832267173976985071561791138510715
	ci_{13} = 175509574710531250	cr_{13} = +1584.483175231209421948555179007853904973
	ci_{14} = -68185176240903550	cr_{14} = -615.5690635800040216043210605232945033063
	ci_{15} = 20014523360265510	cr_{15} = +180.6891480246395236371261503536635811768
	ci_{16} = -4031052737981102	cr_{16} = -36.39194757513992347706890622822427643121
	ci_{17} = 695097885157727	cr_{17} = +6.275275328925558268476956013699105997754
	num = 19	
Points (N) = 20	den = 2688996956405760000	
Segments (N-1) = 19	ci_{1} = 69028763155644023	cr_{1} = +0.4877456245656414838932399711855527103437
Error»O (h^{19})	ci_{2} = -353947208843214156	cr_{2} = -2.500931416824664874149907360722550996191
	ci_{3} = 5438538991957452489	cr_{3} = +38.42780134095441222912798927464052773436
	ci_{4} = -31293644979684279096	cr_{4} = -221.1156294534241283897454235960008639451
	ci_{5} = 128605896878516520180	cr_{5} = +908.7076260428078658921584440557461864750
	ci_{6} = -390165018822674369064	cr_{6} = -2756.840367547146254441527874975328030624
	ci_{7} = 918778256758909425228	cr_{7} = +6491.932553822166738924756753801952128196
	ci_{8} = -1717150274758012947672	cr_{8} = -12133.09488606171610139955772962061897389
	ci_{9} = 2590121803284253347498	cr_{9} = +18301.36480637014593133263836498834201619
	ci_{10} = -3180795372743080898560	cr_{10} = -22474.96485191227391420225755605998238030
	ci_{11} = 3195943710049002658134	cr_{11} = +22582.00045421255508488746450211900589583
	ci_{12} = -2627210173545633198888	cr_{12} = -18563.42498955016870428622513715810293701
	ci_{13} = 1761071043128578083252	cr_{13} = +12443.43164455182698901301793847701099491
	ci_{14} = -954110145180024206808	cr_{14} = -6741.581731892818002066630144404464008849
	ci_{15} = 412501439151154330380	cr_{15} = +2914.665754902132920269066509970278885252

Table 6: Continued

	ci_{16} = -1389495891127597l2968	cr_{16} = -981.794414773618525678039684259456118817l
	ci_{17} = 35595226518134779191	cr_{17} = +251.50988075068246950280410848279971237 80
	ci_{18} = -6365379507657675444	cr_{18} = -44.976700459770281516616045798088985734 41
	ci_{19} = 957599291114022281	cr_{19} = +6.766235449922523319821652091802950500339
	num = 7	
Points (N) = 22	den = 14983859404800000	
Segments (N-1) = 21	ci_1 = 1022779523247467	cr_1 +0.4778112547184455679924133079916924555258
Error»O (h^{21})	ci_2 = -5966218027482930	cr_2 = -2.787234254146951324175841749019345416756
	ci_3 = 98462673861087480	cr_3 = +45.99874427591187804896400625362066397811
	ci_4 = -636505825186027230	cr_4 = -297.3560186286105781653736836856735533910
	ci_5 = 2937368924847356265	cr_5 = +1372.248759044328713574769806959153989832
	ci_6 = -10101722474707772328	cr_6 = -4719.215217696328173705208737223935647803
	ci_7 = 27170960245549634640	cr_7 = +12693.44009313908338147729814982840594999
	ci_8 = -58577829795256960440	cr_8 = -27365.76722252499515658028820321249254545
	ci_9 = 103027050438855916590	cr_9 = +48131.08115796670025959799374211490732740
	ci_{10} = -149446538576972018620	cr_{10} = -69816.84369673698890925674684558022582227
	ci_{11} = 180038500415174725632	cr_{11} = +84108.47091254089177076793175864383294724
	ci_{12} = -180712201906578844740	cr_{12} = -84423.20360673035518924412058295396319601
	ci_{13} = 151179046691155956690	cr_{13} = +70626.21840265571689075329677241793763043
	ci_{14} = -105098354746771143240	cr_{14} = -49098.73106468978837117819030111459044755
	ci_{15} = 60350973167958502320	cr_{15} = +28194.12547613585548957753125006150618309
	ci_{16} = -28329147803950914648	cr_{16} = -13234.50983290264692006301759503279345666
	ci_{17} = 10710818043854933655	cr_{17} = +5003.766004569320689372409667099014129692
	ci_{18} = -3183966521788733730	cr_{18} = -1487.451600445568010859823841371125356490
	ci_{19} = 723790940733103880	cr_{19} = +338.1329501469220272645360159432774124584
	ci_{20} = -116321026020880590	cr_{20} = -54.34161921496169122944278842463475168232
	ci_{21} = 15512151960713877	cr_{21} = +7.246802094940405603664837718806196149286

Table 7: Set of points

x	0.0	0.1	0.2	0.3	0.4	0.5
y = f(x)	1.0000	1.1052	1.2214	1.3499	1.4918	1.6487

For ab>0. Therefore, it can be used only when a is positive and b is ∞ or when a is -∞ and b is negative. For cases where the limits are from -∞ to ∞, the integral can be implemented in three steps. For example:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-A} f(x)dx + \int_{-A}^{A} f(x)dx + \int_{A}^{\infty} f(x)dx \quad (22)$$

Where, A is a positive number. One problem with using Eq. 21 to evaluate an integral is that the transformed function will be singular at one of the limits (Sun and Wu, 2005). The semi-open or semi-closed integration formulas can be used to circumvent this dilemma as they allow evaluation of integral without employing data at the end points of integration interval.

Analysis of error: Let's solve the example problem presented in Eq. 23 using Simpson's 1/3 rule (Sun and Wu, 2008). The exact solution is $I_{exact} = e^{0.8}-e^{0.0}$ = 1.2255409:

$$I = \int_{0}^{0.8} e^{x}dx \quad (23)$$

Solving the problems for two increments of h = 0.4, the minimum permissible number of increments for Simpson's 1/3 rule and one interval yields:

$$I(h=0.4) = \frac{0.4}{3}\left[1+4(1.49182470)+2.22554093\right]$$
$$=1.22571196 \quad (23)$$
$$Error = \left|1.22571196\text{-}1.2255409\right| = 0.0001710$$

Breaking the total range of integration into four increments of h = 0.2 and two intervals and applying the composite rule yields:

$$I(h=0.2) = \frac{0.2}{3}[1+4(1.22140276)+2(1.49182470)$$
$$+4(1.82211880)+2{,}225540938]$$
$$=1.22555177 \quad (23)$$
$$Error = \left|1.22555177\text{-}1.2255409\right| = 0.0000108$$

The global error of Simpson's 1/3 rule is O (h^4). Thus, for successive increment havings:

$$Ratio = \frac{0.0001710}{0.0000108} = 15.775$$
$$Ratio = \frac{Error(h)}{Error(h/2)} = \frac{O\left[(h)^4\right]}{O\left[(h/2)^4\right]} = 2^4 = 16 \cong 15.775 \quad (24)$$

Mixing rules: To estimate the error of integration in discrete function, we can apply different rules or mix several integration formulas of Newton-cotes (Berriochoa *et al*., 2007). The difference in the result of each formula provides an approximation of the error. For example, calculate the integral of the points in Table 7.

Using trapezoidal rule:

$$I = \frac{0.1}{2}[1.0000 + 2(1.1052) + 2(1.2214) + 2(1.3499) + 2(1.4918) + 1.6487] = 0.6493 \quad (25)$$

Using Simpson's 1/3 and 3/8 rule:

$$I = \frac{0.1}{3}[1.0000 + 2(1.1052) + 1.2214] + \frac{3(0.1)}{8}[1.2214 + 3(1.3499) + 3(1.4918) + 1.6487] = 0.6487 \quad (26)$$

Using closed rule with N = 6:

$$I = \frac{5(0.1)}{288}[19(1.0000) + 75(1.1052) + 50(1.2214) + 50.(1.3499) + 75(1.4918) + 19(1.6487)] = 0.6487 \quad (27)$$

Using open rule with N = 6:

$$I = \frac{5(0.1)}{24}[0(1.0000) + 11(1.1052) + 1.2214 + 1.3499 + 11(1.4918) + 0(1.6487)] = 0.6487 \quad (28)$$

Then, estimation of the error is |0.6487-0.6493| = 0.0006 and the best result is 0.6487.

CONCLUSION

Newton-Cotes Integration is very important. To improve the error, formulas of high order should be applied. To estimate the error of integration for set of points proposed to apply different rules or mix various integration formulas. The difference in the result of each formula provides an approximation of the error of integration. And we can choose the best result as the average of all values or the value that appeared more times using the various formulas of integration. Computational routines to generate Newton-Cotes integration rules were presented with number of points until one hundred. The equations of Newton-Cotes shown in the article can be used in many scientific applications.

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