A Tabu Search Method for Finding Minimal Multi-Homogeneous Bézout Number

Hassan M.S. Bawazir and Ali Abd Rahman

Department of Mathematics, Faculty of Science, University Technology Malaysia, Malaysia

Abstract: Problem statement: A homotopy method has proven to be reliable for computing all of the isolated solutions of a multivariate polynomial system. The multi-homogeneous Bézout number of a polynomial system is the number of paths that one has to trace in order to compute all of its isolated solutions. Each partition of the variables corresponds to a multi-homogeneous Bézout number. It is a crucial problem to find a partition with the minimum multi-homogeneous Bézout number since the size of the space of all the partitions increases exponentially. **Approach:** This study presented a new method by producing the Tabu Search Method (TSM) as a powerful technique for finding minimum multi-homogeneous Bézout number. **Results:** A comparison is made between the new method and some recent methods. It is shown that our algorithm is superior to the latter, besides being simple and efficient in the implementation. **Conclusion:** Furthermore the present study extended the applicability of the Tabu search method.

Key words: Multi-homogeneous Bézout number, polynomial system, homotopy method, local search method, Tabu search method

INTRODUCTION

The development of homotopy continuation methods started around the mid seventies with the study of Garcia and Zangwill (1979). Recently these methods have evolved to becoming reliable and efficient numerical algorithms for approximate all of the isolated solutions of polynomial systems. For a survey (Li, 1987; Watson, 1986). Consider a polynomial system of equations:

$$F(x) = (f_1(x), f_2(x),\ldots,f_n(x)) \qquad (1)$$

where, $x = (x_1, x_2,\ldots,x_n) \in \mathbb{C}^n$. The classical homotopy method for polynomial system is based on the classical Bézout number, i.e., the total degree TD, $TD = \prod_{i=1}^{n} d_i$, where d_i is the degree of the ith equation f_i. TD is an upper bound of the number of the isolated solutions of (1) and hence the number of curves one has to trace in the homotopy. However, TD is often far larger than the number of isolated solutions of the system (1). Hence a homotopy goes through exhaustive computations, including tracing unnecessary curves.

Morgan and Sommese (1987) proposes the multi-homogeneous Bézout theory. It is shown that the multi-homogeneous Bézout number also gives an upper bound for the number of isolated solutions of a polynomial system. Each partition of the variables usually gives a different multi-homogeneous Bézout number. It is desired to find a partition whose multi-homogeneous Bézout number is the smallest among all possible variable partitions. In fact the minimal multi-homogeneous Bézout number is usually smaller (sometimes far smaller) than the Bézout number, TD. Thus a smaller number of paths is followed in the multi-homogeneous homotopy method.

Wampler (1992) presents an exhaustive search method on finding the optimal bound. However, numerical experiments show that while it works well for small systems, it is costly when n increases.

Li and Bai (2000) provides a local search method for minimizing multi-homogeneous Bézout numbers; as with any other local search methods, it gives a local minimum rather than the (global) minimum over all possible homogenizations.

Li et al. (2003) presents the so-called fission and assembly operations to generate the partitions from each other in order to minimize homogeneous Bézout numbers, but the search technique is still local in nature and it only works for small systems.

Yan et al. (2008) provides a genetic algorithm for finding minimal multi-homogeneous Bézout numbers; the algorithm depends heavily on random choices from the population space and computes their fitness functions and keep the minimum ones, repeating this

---

Corresponding Author: Hassan M.S. Bawazir, Department of Mathematics, Faculty of Science,
University Technology Malaysia, Malaysia

choice until some stopping criterion is satisfied. This method depends on non-controlled moves in search of feasible solutions, so in sometimes it revisits a large number of candidates and sometimes it encounters an infinite cycling loop.

The computation of minimum Bézout number is a NP-hard problem (Malajovich and Meer, 2007); consequently the topic of minimizing Bézout number is very important because it gives the number of the paths that need to be traced in a homotopy method.

In this study we will present a heuristic technique based on the Tabu search. The Tabu search exhibits several strengths, listed as follows (Glover and Kochenberger, 2003):

- Converges near the optimal solution
- Can be considered as a controlled random walk in the space of feasible solutions
- Uses the short term memory to prevent the reversal of recent moves
- Uses the long term frequency memory to reinforce attractive components
- Uses a 'Tabu list' to prevent cycling back to previously visited solutions. Tabu list records the recent search history, a key idea that can be linked to Artificial Intelligence concepts

The multi-homogeneous Bézout number: Consider the multivariate polynomial system (1). Let $Z = \{z_1, z_2,...,z_m\}$ be a m-partition of the unknowns $X = \{x_1, x_2,...,x_n\}$ where

$$z_j = \left\{ z_{j1}, z_{j2},...,z_{jk_j} \right\}, \qquad j = 1, 2,..., m$$

Define the degree matrix of the system $F(x) = 0$ as the following:

$$D = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ . & . & . & . \\ . & . & . & . \\ d_{n1} & d_{n2} & \cdots & d_{nm} \end{pmatrix}$$

where, d_{ij} is the degree of polynomial f_i w.r.t. the variable z_j. The degree polynomial of F with respect to the partition Z is defined as:

$$f_D(y) = \prod_{i=1}^{n} \sum_{j=1}^{m} d_{ij} y_j$$

The m-homogeneous Bézout number of F w.r.t. the partition Z equals the coefficient of $y^k = y_1^{k_1} y_2^{k_2}...y_m^{k_m}$ in the degree polynomial $f_D(y)$, and denoted by B_m, where $k = (k_1, k_2,...,k_m)$, with $k_j = \#(z_j)$, $j = 1, 2,...,m$ and $\sum_{j=1}^{m} k_j = n$.

The following well-known example shows the significance of the difference between the minimal multi-homogeneous Bézout number and the classical Bézout number (Li *et al.*, 2003).

Example: Consider the matrix eigenvalue problem:

$$Ax = \lambda x, x = (x_1, x_2,...,x_n) \in \mathbb{C}^n, \lambda \in \mathbb{C}.$$

One can view it as a polynomial system of n + 1 variables $(x_1, x_2,...,x_n, \lambda) \in \mathbb{C}^{n+1}$:

$$Ax = \lambda x$$

$$\eta^T x = 1$$

where, $\eta \in \mathbb{C}^n$ is a randomly chosen vector. Clearly, the classical Bézout number of this system is $TD = 2^n$. But we all know that this eigenvalue problem only has n solutions, counting multiplicities. Since $2^n \gg n$, the homotopy method will be very costly. By taking the partition $Z = \{z_1 = X, z_2 = \{\lambda\}\}$ we find the 2-homogeneous Bézout number exactly.

We seek to minimize Bézout number and since each partition Z of the set of unknowns X gives one number then the search space will be the space of all partitions of set of n elements. The total number of all possible partitions is denoted by B(n) and called Bell number. It is the number of all possible ways of putting n distinct balls into n identical boxes, where some of the boxes could be empty. The following recursive relationship holds for Bell numbers (Li *et al.*, 2003):

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(k), B(0) = 1$$

There is an estimation of Bell number given by (Li and Bai, 2000):

$$\left(\frac{n}{2} \right)^{\frac{n}{2}} < B(n) < n!$$

This means that Bell number increases exponentially as n grows. For example, B(4) = 15, B(5) = 52, B(10) = 115,957 and B(15) = 1,382,958,545.

It is not necessary to compute Bézout number by expanding the degree polynomial because we need just the coefficient of one desired monomial. Wampler

(1992) gives a recursive relation for computing Bézout number by letting a partition, say m-partition Z, fixed, $k = (k_1, k_2, \ldots, k_m)$ the cardinalities of the sets in Z, D the corresponding degree matrix whose entries are d_{ij}:

$$b(D,k,i) = \begin{cases} 1, & \text{if } i = n+1 \\ \sum_{j1, k_j \neq 0}^{m} d_{ij} \times b(D, k - e_j, i+1) & \text{otherwise} \end{cases} \quad (2)$$

where, e_j is the jth row of the identity matrix of degree m and the m-homogeneous Bézout number is b (D,k,1).

MATERIALS AND METHODS

The method used by Wampler (1992) is exhaustive, in the sense that it searches over all the partitions. It only works well in small systems. The local search method of Li and Bai (2000) reduces the number of visited solutions compared to Wampler method but sometimes fails to obtain the optimal solution. The a priori cost shown in Li *et al.* (2003) shows that the assembly method reduces the number of visited solutions to n^3, i.e., in a polynomial time and the fission method reduces the number of visited solutions to 2^n-n i.e., much less than that of Wapler, but still exhibits an exponential growth.

A Genetic Algorithm (GA) for minimizing multi-homogeneous Bézout number presented by Yan *et al.* (2008) is heuristic I nature; this algorithm shows some attractive results compared to local search methods, but it depends heavily on non-controlled random walk through the feasible solutions which makes it costly, especially for large-scale systems.

Tabu Search Method (TSM): Tabu search method is a heuristic based on a good controlled random walk through attractive feasible solutions and converges near the optimal solution. TSM is well-suited for hard optimization problems.

The crucial concept in TSM is the definition of the neighborhood of a fixed feasible solution, because each problem has its different nature. In our method let P = $\{z_1, z_2, \ldots, z_m\}$ be a given feasible solution, where $z_i \subseteq X$, $z_i \cap z_j$ for $i \neq j$ and $\sum_{i=1}^{m} \#(z_i) = n$. The neighborhood of P can be given in many ways. From a given feasible solution such P we will generate another feasible solution by one of the following: firstly, split one of z_i's or more to one or more parts, secondly, merge two or more of z_i's into one part, thirdly, move one element or more from one set z_i to another set z_j. Each type of generation can give so many feasible solutions. The

neighborhood of P, denoted by N(P), will generally be defined as a subset of the set of all possible cases.

In our method we will focus on a simple type of neighborhoods, as in the following definition.

Definition 1: Let P = $\{z_1, z_2, \ldots, z_m\}$ be a partition of the set of the variables X = $\{x_1, x_2, \ldots, x_n\}$ of the polynomial system. let $\dot{N}(P)$ be the set of all feasible solutions generated from P by splitting one z_i's at a time into two parts, or merging two z_i's at a time into one part. We define a typical neighborhood of the partition p as N (P) which has $\dot{n}$ elements chosen randomly from $\dot{N}(P)$ where:

$$\dot{n} = \begin{cases} n, & \text{if } \#(\dot{N}(P)) > n \\ \#(\dot{N}(P)), & \text{otherwise} \end{cases}$$

In the following we establish our algorithm for minimizing multi-homogeneous Bézout number using TSM, where L the tabu list for storing partitions and L_B for storing the corresponding Bézout numbers.

Algorithm 1: Finding the minimum multi-homogeneous Bézout number using TSM.

S0: Input criterion numbers $M, \dot{M} > 0$. Set $i = 0$, $L = \emptyset$ and $L_B = \emptyset$. Go to step S1.
S1: Choose random partition P, add P to the tabu list L, compute B(P) and add it to L_B. Go to step S2.
S2: If B(P)≤M or $i \geq \dot{M}$ declare the result: the partitions in L whose Bézout number is the minimum over all the values in L_B and stop. It is considered as an approximation of the minimum Bézout number over all partitions. Otherwise let $i := i+1$ and go to step S3.
S3: Generate $\dot{N}(P)$, then N(P) as in Definition 1, compute $B(N(P)) = \{(B(s) : s \in N(P)\}$. Go to step S4.
S4: Let $Y = \{y \in N(P) : B(y) = \min(B(N(P)))\}$. Pick $\dot{P} \in Y$ such that $\dot{P} \notin L$, go to step S5.
S5: Let $P := \dot{P}$, add P to L and add B(P) to L_B. Go to step S2.

Note that B(P) is the multi-homogeneous Bézout number of the partition p computed using (2).

We note that choosing the neighborhood with the number of elements not more than the size of the system decreases the computation cost. From other viewpoint the randomness and the fact that optimal solution can be reached in so many ways pay off what can arise from decreasing the size of the neighborhood.

Algorithm 1 generates #(Ṅ(P)) feasible solutions and computes fitness function for just random # (N(P)) #(N(P))≤n solutions in step S3; this procedure is repeated no more than n times in one loop and previous work is done Ṁ times in one run of the algorithm; so the number of evaluated solutions is $\dot{M}n^2$ or $O(n^2)$, i.e., polynomial time.

RESULTS

We implement our method on twenty multivariate polynomial systems with different sizes; these systems are cited in (Li and Bai, 2000). Table 1 provides basic information about the used systems. Table 2 and 3 summarize the implementation of Local Search Method (LSM) (Li and Bai, 2000), Fission Method (FM), Assembly Method (AM) (Li *et al.*, 2003), Genetic Algorithm (GA) (Yan *et al.*, 2008) and Tabu Search Method (TSM) in two aspects, firstly the convergence to the global optimal solution (MB) and secondly, how many feasible solutions needed to be visited to arrive at the Optimal Solution (OS). The column of percentage shows the percentage of the number of visited solutions to the whole population. The result is the average of 10 times run of the program. Some symbols used are listed below:

TD = Total degree (classical Bézout number)
MB = Minimum Bézout number
nP = The number of all partitions
vP = The number of visited solutions (partitions)
OS = The optimal solution (global or local)
\# = The number of the system as cited by (Li and Bai, 2000)
* = The solution is not global

DISCUSSION

On the one hand, the result in Table 2 shows that LSM, FM and AM may fail to reach the global solution; moreover, each of these approximates a solution through visiting so many feasible solutions. On the other hand, the result in Table 3 shows that GA achieves the optimal solution by visiting a less number of feasible solutions comparing to LSM, FM, and AM. However GA is still inefficient since it goes through exhaustive searching in the space of feasible solutions whereas TSM achieves the optimal solution through a considerably shorter path of the solutions as shown in Table 3.

Table 1: Basic information of the used systems in TSM

Sr. No.	n	TD	MB	NP
1	2	16	10	2
2	4	625	384	15
3	4	256	96	15
4	4	144	62	15
5	4	900	450	15
6	5	16	16	52
7	6	8	8	203
8	8	5764801	645120	4140
9	8	128	16	4140
10	6	64	20	203
11	11	2048	320	678570
12	8	576	193	4140
13	7	4608	1361	877
14	10	3628800	3628800	115975
15	9	362880	362880	21147
16	6	1024	216	203
17	5	108	56	52
18	4	1344	368	15
19	10	64	56	115975
20	8	256	160	4140

Table 2: Numerical results (1)

Sr. No.	n	LSM			FM			AM		
		VP	%	OS	VP	%	OS	VP	%	OS
1	2	1	50.00	10	1	50.00	10	1	50.00	10
2	4	12	80.00	384	41	73.00	384	11	73.00	384
3	4	12	80.00	96	11	73.00	96	11	73.00	96
4	4	11	73.00	62	11	73.00	62	11	73.00	62
5	4	13	87.00	450	12	80.00	450	11	73.00	450
6	5	30	58.00	16	24	46.00	16	21	40.00	16
7	6	72	35.00	8	58	29.00	8	36	18.00	8
8	8	260	6.28	645120	161	3.89	645120	85	2.05	645120
9	8	283	6.84	16	160	3.86	16	50	1.21	16
10	6	70	34.00	20	46	23.00	20	36	18.00	20
11	11	2617	0.39	576*	1145	0.17	320	221	0.03	320
12	8	191	4.61	193	170	4.11	193	85	2.05	193
13	7	130	15.00	1361	101	12.00	1361	57	7.00	1361
14	10	6147	5.30	3628800	615	0.53	362880	166	0.14	3628800
15	9	934	4.42	362880	320	1.51	362880	121	0.57	362880
16	6	103	51.00	344*	46	23.00	216	36	18.00	216
17	5	27	52.00	56	27	52.00	56	21	40.00	56
18	4	12	80.00	368	12	80.00	368	11	73.00	368
19	10	1752	1.51	64*	946	0.82	44	166	0.14	48*
20	8	278	7.00	160	Error report			Error report		

Table 3: Numerical results (2)

Sr. No.	n	GA			TSM		
		VP	%	OS	VP	%	OS
1	2	2	100.00	10	1	50.00	10
2	4	11	73.00	384	5	33.00	384
3	4	11	73.00	96	8	53.00	96
4	4	11	73.00	62	7	47.00	62
5	4	12	80.00	450	6	40.00	450
6	5	28	54.00	16	13	25.00	16
7	6	68	33.00	8	23	11.00	8
8	8	239	5.77	645120	48	1.16	645120
9	8	192	4.64	16	42	1.02	16
10	6	68	33.00	20	26	13.00	20
11	11	613	0.09	320	207	0.03	320
12	8	180	4.35	193	63	1.52	193
13	7	125	14.00	1361	46	5.00	1361
14	10	3015	2.60	3628800	118	0.10	3628800
15	9	603	2.85	362880	101	0.48	362880
16	6	87	43.00	216	22	11.00	116
17	5	24	46.00	56	14	27.00	56
18	4	11	73.00	368	7	47.00	368
19	10	504	0.43	44	155	0.13	44
20	8	201	5.00	160	77	2.00	160

CONCLUSION

We have presented a heuristic method based on Tabu search method. Two aspects of the performance are clear in the numerical results. Firstly where the local search method, fission method and assembly method may fail to achieve the optimal solution, as shown in Table 2, while our Tabu search method obtain the optimal solution with less number of visited solutions, see Table 3. As for the heuristic method genetic algorithm, one can reach the optimal solution but by costly non-controlled walk through the feasible solutions as shown in Table 3. TSM is easy, competitive and efficient to implement so it can deal well with large scale systems. This study extends also the application fields of Tabu search method.

REFERENCES

Garcia, C.B. and W.L. Zangwill, 1979. Finding all solutions to polynomial systems and other systems of equations. Math. Programm., 16: 159-176. http://www.springerlink.com/content/q15m040467 x3m263/fulltext.pdf

Glover, F. and Gary A. Kochenberger, 2003. Handbook of Metaheuristics. 1st edition, Kluer Academic Publishers, ISBN: 0-306-48056-5, pp: 37-54.

Li, T. and F. Bai, 2000. Minimizing multi-homogeneous Bézout number by a local search method. Math. Comp., 70: 767-787. http://www.ams.org/mcom/2001-70-234/S0025-5718-00-01303-X/S0025-5718-00-01303-X.pdf

Li, T., Z. Lin and F.S. Bai, 2003. Heuristic methods for computating the minimal multi-homogeneous Bézout number. Appl. Math. Comp., 146: 237-256. DOI: 10.1016/S0096-3003(02)00540-4

Li, T.Y., 1987. Solving polynomial systems. Math. Intel., 9: 33-39. http://www.springerlink.com/content/gj51ug83835 7h345/fulltext.pdf

Malajovich, G. and K. Meer, 2007. Computing multi-homogeneous Bézout numbers is hard. Theory Comput. Systems, 40: 553-570. DOI: 10.1007/s00224-006-1322-y

Morgan, A. P. and A. J. Sommese, 1987. A homotopy for solving general polynomial systems that respects m-homogeneous structures. Applied Math. Comput., 24: 101-113. DOI: 10.1016/0096-3003(87)90063-4

Wampler, C.W., 1992. Bézout number calculations for multi-homogeneous polynomial systems. Applied Math. Comput., 51: 143-157. DOI: 10.1016/0096-3003(92)90070-H

Watson, L.T., 1986. Numerical linear algebra aspects of globally convergent homotopy methods. SIAM. Rev., 28: 529-545. http://www.jstor.org/stable/2031103

Yan, D., J. Zhang, B. Yu, C. Luo and S. Zhang, 2008. A genetic algorithm for finding minimal multi-homogeneous Bézout number. Proceeding of the 7th IEEE/ACIS International Conference on Computer and Information Science, May 2008, IEEE Computer Society, USA., pp: 301-305. DOI: 10.1109/ICIS.2008.38