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Asymptotic Distribution of Coefficients of Skewness and Kurtosis

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Abstract: Problem statement: In literature, a classic method which has been used to recognize the distribution so far is the measurement of its skewedness and kurtosis. However, there remains a question: how would these measurements work for skewed normal distribution when the size of the sample is large? **Approach:** This research aimed to determine the asymptotic distribution of skewedness and kurtosis measures in skewed normal distribution. In conducting this research, two groups of inferential findings will help. First, skewed normal distribution which has already been studied by a lot of researchers and we apply its characteristics. Second, there is the U-statistics theory which guides us to the determining of asymptotic distribution measures for skewedness and kurtosis. The combination of these two will solve the problem of this study. **Results:** Asymptotic distribution of measures for skewdness and kurtosis falls in the normal families. With the size of large samples, the values of expectation of these measures are also determined. By letting zero for skewedness parameter, asymptotic distribution for normal distribution can also be obtained. **Conclusion:** The findings of this study show new characteristics for skew normal distribution and this results in a new way for skew normal distribution recognition.

Key words: Central moments, eigenvalues, limiting distribution, U-statistics

INTRODUCTION

It is well know the criteria of skewness and kurtosis are made function of sample variance, 3rd and 4th central moments of sample. The U-statistics help to find best estimators of central moments.

In the theory of U-statistics, we consider a functional θ defined on a set F of distribution functions on $\Re$: $\theta = \theta(F)$, $F \in \mathcal{F}$. The $\theta \in \theta(F)$ estimated by using a sample from the random variables $X_1, X_2, \ldots, X_n$ which are independently and identically distributed with distribution function Halmos[4] proved that the functional θ admits an unbiased estimator if and only if there is a function h of k variables such that:

$$\theta = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h(x_1, \ldots, x_k)\, dF(x_1) \ldots dF(x_k) \qquad (1)$$

A functional satisfying Eq. 1 for some function h is called a regular statistical functional of degrees k and the function h is called the kernel of the functional. If a functional can be written as a regular statistical functional then optimal unbiased estimators can be constructed.

For a distribution function F let:

$$\mu'_r = \int_{-\infty}^{+\infty} x^r dF$$

The rth moment about 0 and let:

$$\mu_r = \int_{-\infty}^{+\infty} (x - \mu'_1)^r dF$$

the rth central moment. Heffernan,[5] obtained an estimator of the rth central moment of a distribution, which unbiased for all distributions for which the first r moments exits. There is a unique symmetric unbiased estimator of μ_r:

$$U_r(x_1, \ldots, x_n) = \frac{(n-r)!}{n!} \sum h_r(x_{i_1}, \ldots, x_{i_r})$$

where, the sum extends over all $\frac{(n-r)!}{n!}$ permutations $(i_1, \ldots, i_r)$ of r distinct integers chosen from $1, 2, \ldots, n$ and:

$$h_r(x_{i_1}, \ldots, x_{i_r}) = \sum_{j=0}^{r-2} (-1)^j \frac{1}{r-j} \sum x_{i_1}{}^{r-j} x_{i_2} \ldots x_{i_{j+1}} + (-1)^{r-1}(r-1) x_1 \cdots x_r \qquad (2)$$

where, the second summation is over $i_1, \ldots, i_{j+1} = 1$ to r with $i_1 \neq i_2 \neq \ldots \neq i_{j+1}$ and $i_1 < i_2 < \cdots < i_{j+1}$.

Consider a symmetric kernel h satisfying:

$E_F(h^2(X_1,\ldots,X_k)) < \infty$

We shall make use of the function h_c and $\tilde{h}_k$. $h_k = h$ and for $1 \le c \le k-1$:

$h_c = E_F(h(x_1,\ldots,x_c,X_{c+1},\ldots,X_k))$

That $\tilde{h} = h - \theta$, $\tilde{h}_c = h_c - \theta$, Define $\xi_0 = 0$ and, for $1 \le c \le k-1$:

$\xi_c = \mathrm{var}_F(h_c(X_1,\ldots,X_c))$

The following theorem were established by Hoeffding[6].

Theorem 1: If $E_F h^2 < \infty$ and $\xi_1 > 0$, then:

$$\sqrt{n}(U_n - \theta) \xrightarrow{D} N(0, k^2\xi_1)$$

Heffernan obtained an estimator, U_r, of rth central moment of a distribution, which unbiased for all distributions for which the first r moments exits. But the form of U_r does not have a presentation of the role of sample moments on estimation of μ_r obviously. Abbasi[1] has been tried to establish a connection between U-statistics and sampling moments and also to studied their asymptotic distribution. This study was done for r = 3, 4.

In this article we study two measured that indicate the value of skewness and the value of kurtosis in skew normal distribution.

The univariate skew normal distribution has been considered by several authors. Azzalini[3] introduced this distribution. A random variable X is said to be skew normal distribution if:

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Phi(\lambda x), \qquad x \in R$$

where, Φ cumulative distribution function of standard normal distribution. The skew normal distribution is allows for continuous variation from normality to non-normality, which is useful in many practicat situations[2,8,10] generated skewed probability density function of the form $2f(u)G(\lambda u)$, where , f is taken to be a normal pdf while the cumulative distributive function G is taken to come from one of normal, Student's t, Cauchy, Laplace, logistic or uniform distribution. In particular, expressions for the nth moment and characteristic function were derived. In skew normal distribution, the nth moment of X about zero turns out to be:

$$E(X^n) = \frac{2^{\frac{n+2}{2}}\lambda}{\pi} \Gamma(\frac{n+2}{2}) \left(1+\lambda^2\right)^{\frac{-n}{2}} \sum_{k=0}^{\frac{n-1}{2}} \frac{\left(\frac{1-n}{2}\right)_k}{\left(\frac{3}{2}\right)_k} \left(-\lambda^2\right)^k \tag{3}$$

if n is odd and:

$$E(X^n) = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \Gamma(\frac{n+1}{2}) \tag{4}$$

for n even, where, $(c)_k = c(c+1)\cdots(c+k-1)$.

In point estimation, estimator be a UMVU (uniform minimum variance unbiased) estimator is a advantage. The U-statistics admits UMVU estimator[7,9].

MATERIALS AND METHODS

The form of U-statistics for 2nd, 3rd and 4th central moment equal to:

$$U_2 = S^2 = \frac{n}{n-1}(\overline{x^2} - \overline{x}^2) = \frac{n}{n-1} M_2$$

$$U_3 = \frac{n^2}{(n-1)(n-2)}\left[\overline{x^3} - 3\overline{x^2}\,\overline{x} + 2\overline{x}^3\right] = \frac{n^2}{(n-1)(n-2)} M_3$$

$$U_4(x_1,\ldots,x_n) = \frac{n^3}{(n-1)(n-2)(n-3)} M_4 + \frac{-2n^2+3n}{(n-1)(n-2)(n-3)} \overline{x^4} - \frac{8n^2-12n}{(n-1)(n-2)(n-3)} \overline{x^3}\,\overline{x} + \frac{-6n^2+9n}{(n-1)(n-2)(n-3)} \overline{x^2}^2$$

Where:

$$M_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

$$M_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3$$

$$M_4 = \tfrac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^4$$

Are sample central moments. Obviously $U_r \to M_r$ as $n \to \infty$ for r = 2,3,4. Also by theorem 1 we can show that:

$$\sqrt{n}(S^2 - \sigma^2) \sim AN(0\,,\mu_4 - \sigma^4),) \quad (5)$$

$$\sqrt{n}(U_3 - \mu_3) \sim AN(0\,,9\xi_{3,1}) \quad (6)$$

$$\sqrt{n}(U_4 - \mu_4) \sim AN(0\,,16\xi_{4,1}) \quad (7)$$

Where:

$$\begin{aligned}\xi_{3,1} = \frac{1}{36}&[4\,\mathrm{var}(X^{\ 3}) + 36\mu'^2\,\mathrm{var}(X^{\ 2})\\ &+(12\mu'^2 - 6\mu'_2)^2\,\mathrm{var}(X\ ) - 2(6\mu')\mathrm{cov}(X^{\ 3}, X^{\ 2})\\ &+2(12\mu'^2 - 6\mu'_2)\mathrm{cov}(X^{\ 3}, X\ )\\ &-2(6\mu')(12\mu'^2 - 6\mu'_2)\mathrm{cov}(X^{\ 2}, X\ )]\end{aligned}$$

and

$$\begin{aligned}\xi 4{,}1 = \frac{1}{16}&[\mathrm{var}(X^4) + 16\mu_1'^2\,\mathrm{var}(X^3) + 36\mu_1'^4\,\mathrm{var}(X^2)\\ &+(-4\mu'_3 + 12\mu'_2\mu'_1 - 3\mu_1'^3)^2\,\mathrm{var}(X)\\ &-2(4\mu'_1)\mathrm{cov}(X^4, X^3) + 2(6\mu_1'^2)\mathrm{cov}(X^4, X^2)\\ &+2(4\mu'_3 + 12\mu'_2\mu'_1 - 3\mu_1'^3)\mathrm{cov}(X^4, X)\\ &-2(24\mu_1'^3)\mathrm{cov}(X^3, X^2)\\ &-2(4\mu'_1)(-4\mu'_3 + 12\mu'_2\mu'_1 - 3\mu_1'^3)\mathrm{cov}(X^3, X)\\ &+2(6\mu_1'^2)\left(-4\mu'_3 + 12\mu'_2\mu'_1 - 3\mu_1'^3\right)\mathrm{cov}(x^2, X)]\end{aligned}$$

RESULTS

Nadarajah *et al*.[10] in skewed normal distribution show that how to compute the value of central moment. For our aims, we have:

$$\mu'_1 = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}},$$

$$\mu'_2 = 1,$$

$$\mu'_3 = 3\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\lambda(1 - \frac{3}{2}\lambda^2)}{(1+\lambda^2)^{\frac{3}{2}}},$$

$$\mu'_4 = 3,$$

$$\mu'_5 = 15\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\lambda(1 - \frac{4}{3}\lambda^2 + \frac{8}{5}\lambda^4)}{(1+\lambda^2)^{\frac{5}{2}}},$$

$$\mu'_6 = 15,$$

$$\mu'_5 = 105\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\lambda(1 - 2\lambda^2 + \frac{16}{5}\lambda^4 - \frac{32}{7}\lambda^6)}{(1+\lambda^2)^{\frac{7}{2}}}$$

$$\mu'_8 = 105,$$

$$\mu_2 = \sigma^2 = 1 - \frac{2\lambda^2}{\pi(1+\lambda^2)},$$

$$\mu_3 = \frac{-\sqrt{2}\,\lambda^3\,(5\pi - 4)}{\pi^{\frac{3}{2}}(1+\lambda^2)^{\frac{3}{2}}}$$

$$\mu_4 = \frac{3\pi^2 + 6\pi^2\lambda^2 + 3\pi^2\lambda^4 - 12\pi\lambda^2 + 28\pi\lambda^4 - 12\lambda^4}{\pi^2(1+\lambda^2)^2}$$

The following values determine the distributions in (1), (2) and (3):

$$\mu_4 - \sigma^4 = \frac{2}{\pi^2\left(1+\lambda^2\right)^2}\{\pi^2 + 2\pi^2\lambda^2 + \pi^2\lambda^4 - 4\pi\lambda^2 + 16\pi\lambda^4 - 8\lambda^4\}$$

$$\begin{aligned}\xi_{3,1} = \frac{1}{9\pi^3(1+\lambda^2)^3}&\{\lambda^6\left(15\pi^3 - 302\pi^2 + 798\pi - 288\right)\\ &+\lambda^4\left(45\pi^3 + 84\pi^2 - 72\pi\right)\\ &+\lambda^2\left(-18\pi^2 + 45\pi^3\right) + 15\pi^3\}\end{aligned}$$

$$\begin{aligned}\xi_{4,1} = \frac{1}{2\pi^3(1+a^2)^4}&(-32\pi^2 + 48\pi^3\lambda^2 + 72\pi^3\lambda^4\\ &+60\pi^2\lambda^7\sqrt{1+\lambda^2} - 120\pi\lambda^3\sqrt{1+\lambda^2}\\ &+180\pi^2\lambda^3\sqrt{1+\lambda^2} + 60\pi^2\sqrt{1+\lambda^2}\\ &+120\pi\lambda^7\sqrt{1+\lambda^2} + 180\pi^2\lambda^5\sqrt{1+\lambda^2}\\ &-48\lambda^5\sqrt{1+\lambda^2} + 967\pi^2\lambda^8 + 627\pi\,\lambda^7\\ &-72\lambda^6 + 48\pi^3\lambda^6 - 48\lambda^7\sqrt{1+\lambda^2}\\ &+12\pi^3 - 480\pi\,\lambda^5 + 288\pi\,\lambda^3\\ &-675\pi\,\lambda^8 + 498\pi\,\lambda^6 - 653\pi^2\lambda^6 - 64\pi\,\lambda^2\\ &+108\lambda^8 - 269\pi^2\lambda^2 + 405\pi^2\lambda^4\\ &-507\pi\,\lambda^4 + 12\pi^3\lambda^8)\end{aligned}$$

Asymptotic distribution of skewness: By Stlusky's theorem and tending of S^2 to σ^2 with probability one, we have:

$$\frac{\sqrt{n}(M_3 - \mu_3)}{S^3} \sim AN(0\,,\frac{9\xi}{\sigma^6})$$

Therefore the approximate of expectation of skewness is:

$$E(\frac{M_3}{S^3}) \cong \frac{\dfrac{-\sqrt{2}\,\lambda^3\,(5\pi-4)}{\pi^{\frac{3}{2}}(1+\lambda^2)^{\frac{3}{2}}}}{\left(1-\dfrac{2\lambda^2}{\pi(1+\lambda^2)}\right)^{\frac{3}{2}}}$$

$$= -\sqrt{2}\,(5\pi-4)\frac{\left(\lambda^2\right)^{\frac{3}{2}}}{(1+\lambda^2)^{\frac{3}{2}}}\frac{1}{\left(\pi-2\dfrac{\lambda^2}{(1+\lambda^2)}\right)^{\frac{3}{2}}}$$

$$= -\sqrt{2}\,(5\pi-4)\frac{T^{\frac{3}{2}}}{(\pi-2T)^{\frac{3}{2}}}, \quad (T=\sqrt{\frac{\lambda^2}{(1+\lambda^2)}})$$

Asymptotic distribution of kurtosis: Again use the Stlusky's theorem and tending of S^2 to σ^2 with probability one, then we have:

$$E(\frac{M_4}{S^4}) \cong \frac{\dfrac{3\pi^2+6\pi^2\lambda^2+3\pi^2\lambda^4-12\pi\lambda^2+28\pi\lambda^4-12\lambda^4}{\pi^2(1+\lambda^2)^2}}{\left(1-\dfrac{2\lambda^2}{\pi(1+\lambda^2)}\right)^2}$$

$$= \frac{3\pi^2+6\pi^2\lambda^2+3\pi^2\lambda^4-12\pi\,\lambda^2+28\pi\,\lambda^4-12\lambda^4}{\left(\pi+\pi\lambda^2-2\lambda^2\right)^2}$$

DISCUSSION

In special case, when λ tends to infinity, limiting distributions and approximate of expectation change to:

$$\sqrt{n}(S^2-1+\frac{2}{\pi}) \sim AN(0\,,\,2-\frac{16}{\pi^2})$$

$$\sqrt{n}(U_3+\frac{5\sqrt{2}}{\sqrt{\pi}}) \sim AN(0\,,15+\frac{-302\pi^2+798\pi-288}{\pi^3})$$

$$E(\frac{M_3}{S^3}) \to \frac{-\sqrt{2}\,(5\pi-4)}{(\pi-2)^{\frac{3}{2}}}$$

$$E(\frac{M_4}{S^4}) \cong \frac{28\pi\;-12+3\pi^2}{(\pi-2)^2}$$

CONCLUSION

By these results, for large numbers of observation, the expected values of coefficients of skewness and kurtosis, in skewed normal distribution, are obtained. The results rewrite when the mean and variance of normal distribution are not zero and one respectively.

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