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A Priori Estimation of the Resolvent on Approximation of Born-Oppenheimer

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Abstract: In this study, we estimate the resolvent of the two bodies Shrodinger operator perturbed by a potential of Coulombian type on Hilbert space when h tends to zero. Using the Feschbach method, we first distorted it and then reduced it to a diagonal matrix. We considered a case where two energy levels cross in the classical forbidden region. Under the assumption that the second energy level admits a non degenerate point well and virial conditions on the others levels, a good estimate of the resolvent were observed.

Key words: Distorsion, eigenvalues, estimation, resolvent, resonances

INTRODUCTION

The Born-Oppenheimer approximation technical[1] has instigated many works one can find in bibliography the recent papers like[2-5].

It consists to study the behaviour of a many body systems, in the limit of small parameter h as the particles masses (masses of nuclei) tends to infinity; (see the references therein for more information), we can describe it with a Hamiltonian of type $P = -h^2\Delta_x - \Delta_y + V(x,y)$ on $L^2(IR_x^3 \times IR_y^{3p})$, when h $\to 0$ and V denote the interaction potentials between the nuclei of the molecule and the nuclei electrons.

The idea is to replace the operator $Q(x) = -\Delta_y + V(x,y)$ (in , $L^2(IR_y^{3p})$ x fixed) by the so-called electronic levels which be a family of its discrete eigenvalues: $\lambda_1(x), \lambda_2(x), \lambda_3(x), ...$ and to study the operators P which can be approximativelly given by

$-h^2\Delta_x + \lambda_j(x)$, on $L^2(IR_x^3)$.

Martinez and Messirdi's works, are about spectral proprieties of P near the energy level E_0 such that $\inf_{R^n} \lambda_j \leq E_0$. Martinez in[6], studies the case where $\lambda_1(x)$ admits a nondegenerate strict minimum at some energy level λ_0, the eigenvalues of P near λ_0 admits a complete asymptotic expansion in half-powers of h[2].

Messerdi and Martinez[7] considers the case where λ_2 admits a minimum, such appears resonances for P. He gives an estimation of the resolvent of $O(h^{-1})$ at the neighbourhood of 0.

In this study we try to generalize this work to approximate the resolvent of P where V is a potential of Coulombian type at the neighbourhood of a point $x_0 \neq 0$.

In fact, we estimate the resolvent of the operator F_μ^ς,given by a reduction of the distorted operator P_μ^ς, of P modified by a truncature ς[8]; and we try to have a good evaluation of the order of $O(h^{-1/2})$.

We apply the Feshbach method to study the distorted operator P_μ^ς which allows us to goback to the initial problem and we put the virial conditions on λ_1 and λ_3.

Hypothesis and results

Hypothesis: Let the operator

$$P = -h^2\Delta_x - \Delta_y + V(x,y) \quad (1)$$

on $L^2(IR_x^3 \times IR_y^{3p})$, when h tends to 0. $V(x,y) = V(x, y_1, y_2, y_3, ..., y_p)$ is an interaction potential of Coulombian type

$$V(x,y) = \frac{\alpha}{|x|} + \sum_{j=1}^{p}\left[\frac{\alpha_j^+}{|y_j + x|} + \frac{\alpha_j^-}{|y_j - x|}\right] + \sum_{\substack{j,k=1 \\ j\neq k}}^{p} \frac{\alpha_{jk}}{|y_j - y_k|} \quad (2)$$

where $\alpha, \alpha_j^{\pm}, \alpha_{jk}$ are real constants, $\alpha > 0$ ($\alpha_j^{\pm}$ is the charges of the nuclei).

It is well known that P with domain $H^2(IR_x^3 \times IR_y^{3p})$ is essentially self-adjoint on $L^2(IR_x^3 \times IR_y^{3p})$.

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For $x \neq 0$, $Q(x) = -\Delta_y + V(x,y)$ with domain $H^2(IR_y^{3p})$ is essentielly self-adjoint on $L^2(IR_y^{3p})$

Remark 1.1: The domain of $Q(x)$ is independent of x. To describe our main results we introduce the following assumptions:

(H1) $\forall x \in IR^{3n} \setminus \{0\}$, $\#\ \sigma_{disc}(Q(x)) \geq 3$

Let λ_0 an energy level such that: $\lambda_j \cap [-\infty, \lambda_0[\leq 3$, denoting $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ the first three eigenvalues of *Q(x)*.

(H2) we assume that the first tree eigenvalues λ_j, $\forall j \in \{1,2,3\}$ are simple at infinity:

$$|x| \geq C \Rightarrow \inf_{j,k\in\{1,2,3\}} |\lambda_j(x) - \lambda_k(x)| \geq \frac{1}{C} \qquad (3)$$

and

$$\lim_{j,k\in\{1,2,3\}} \operatorname{dist}(\lambda_j(x) - \lambda_k(x)) \setminus \{\lambda_1(x), \lambda_2(x), \lambda_3(x)\} \rangle 0$$

this means

$\exists \delta_1 \rangle 0, \forall x \neq 0$, and $\lambda \in \sigma(Q(x)) \setminus \{\lambda_1(x), \lambda_2(x), \lambda_3(x)\}$, we have

$$\inf_{1\leq j\leq 3} |\lambda - \lambda_j(x)| \geq \delta_1 \qquad (4)$$

Remark1.2: By Reed-Simon' results[9], the first eigenvalue is automatically simple.

(H3) we suppose that $\exists c \rangle 0$ such that

$$\forall x \in IR^3 \setminus \{0\}, \ \lambda_j \leq c + \frac{\alpha}{x}, \ \ j \in \{1,2,3\} \qquad (5)$$

Remark 1.3: This hypothesis is still true for $\alpha_{\pm} \langle 0$; λ_1 also verifies (H3) and we can see with a simple computation that there exists c_1 such that for all $x \neq 0$

$$\lambda_1(x) \geq -c_1 + \frac{\alpha}{|x|} \qquad (6)$$

(H4) We are in the situation where $\lambda_2(x)$ admits a nondegenerate strict minimum; creating a potential well of the shape Γ : $\begin{cases} \nu_0 = \inf_{x\in IR-\{0\}} \lambda_2(x), & \nu_0 \langle \lambda_0(x) \\ \lambda_2^{-1}(\nu_0) = r_0, \ \lambda_2(x) \rangle 0, & \lambda_2''(r_0) \rangle 0 \end{cases}$

$\exists \delta_2 \rangle 0$ such that

$\forall x \in R^3 \setminus \{0\}$, $\lambda_1(x) + \delta_2 \langle \min\{\lambda_2(x), \lambda_3(x)\}$

we note by

$$K = \{x \in R, \lambda_2(x) = \lambda_3(x)\}$$

and for $\delta \rangle 0$, we also note by: $K_\delta = \{x \in IR, \operatorname{dist}(x,K) \leq \delta\}$

Let $\delta_0 \rangle \delta_1 \rangle 0$ such that

* $K_{\delta_0} \setminus K_{\delta_1}$ is simply connex
* $K_{2\delta_0} \cap U = \varnothing$
* The connex composites of $IR^3 \setminus K_{\delta_1}$ are simply connex

(H5) Virial Conditions

It exists $d \rangle 0$ such that for $j \in \{2,3\}$,

The resonances of *P* are obtained by an analytic distorsion introduced by Hunziker[8] and so they are defined as complex numbers ρ_j ($j = 1, ..., N_0$) such that for all $\varepsilon \rangle 0$ and μ sufficiently small, $\operatorname{Im}\mu \rangle 0$ $\rho_j \in \sigma_{disc}(P\mu)$ [3]. We denote de set of the resonances of *P* by: $\sigma(P) = \bigcup_{\operatorname{Im}\mu\rangle 0, |\mu|\langle\varepsilon} \sigma_{disc}(P_\mu)$

Where P_μ is obtained by the analytic distorsion satisfying: $P_\mu = U_\mu P_\mu U_\mu^{-1}$. So, P_μ can be extended to small enough complex values of μ as an analytic family of type[9].

The analytic distorsion U_μ, for μ small enough associated to *v* is defined on $C_0^\infty(IR_x^3 \times IR_y^{3p})$ by

$$U_\mu \varphi(x,y) = \varphi(x + \mu v(x), y_1 + \mu v(y_1), ..., y_p + \mu v(y_p)) |J|^{1/2}$$

where $J = J(x,y) = \det(1 + \mu Dv(x)) \prod_{j=1}^{p} \det(1 + \mu D(y_j))$ is the Jacobien of the transformation

$\Psi_\mu : (x,y) \to (x + \mu v(x), y_1 + \mu v(x), ..., y_p + \mu v(x))$ and $v \in C^\infty(R^3)$ is a vector field satisfying :

$\exists N \rangle 0$, large enough such that: $\begin{cases} v(x) = 0, \ \text{si } |x| \leq \dfrac{2}{N} \\ v(x) = x, \ \text{si } |x| \geq r_0 - \varepsilon' \end{cases}$

($\varepsilon' \rangle 0$, small enough, $|r_0| \rangle \frac{3}{N} + \varepsilon'$).

Remark 1.4: The distorsion is close to the potential well.

We localise our operator near the well v_0 by introducing a truncate function $\varsigma \in C^\infty(IR^3)$ satisfying:

$$\begin{cases} \varsigma = 1, \ \text{si } |x| \geq \dfrac{2}{N} \\ \varsigma = 0, \ \text{si } |x| \leq \dfrac{3}{2N} \end{cases}$$

fixing $\alpha_0 \rangle v_0$, we set

$Q_\mu^\varsigma(x) = -U_\mu \Delta_y U_\mu^{-1} + \varsigma(x) V_\mu(x,y) + (1-\varsigma(x))\alpha_0$

$V_\mu(x,y) = (x + \mu v(x), y_1 + \mu v(x), ..., y_p + \mu v(x))$

We also denote:

$$P_\mu^\varsigma = -h^2 U_\mu \Delta_x U_\mu^{-1} + Q_\mu^\varsigma(x) \quad (7)$$

With domain $H^2(IR_x^3)$.

Remark1.5: Like in [10], near ν_0, $\sigma(P_\mu)$ and $\sigma(P_\mu^\varsigma)$ coincide up to exponentially small error terms. For this we will study P_μ^ς instead of P_μ.

RESULTS

Here we write the results of our works as following:

Theorem 1.6: Under assumptions (H1) to (H5) and for $\mu \in C, |\mu|$ and *h* small enough, we have $\left\| \left(F_\mu^\varsigma - z\right)^{-1} \right\| = O\left(h^{-1/2}\right)$

where F_μ^ς is the Feshbach reduced operator of P_μ^ς verifying

$F_\mu^\varsigma = -\dfrac{h^2}{(1+\mu)^2}\Delta_x I + M_\mu^\varsigma + \tilde{R}_\mu^\varsigma$ and the error $\tilde{R}_\mu^\varsigma$ is satisfying: $\left\|\tilde{R}_\mu^\varsigma\right\|_{L(H^m \oplus H^m, H^{m-1} \oplus H^{m-1}} = O(h^2)$

We need for our proof the main important theorem for the operator $P_{2,\mu}^\varsigma$ which is the distorsion of the operator $P_{2,\mu}$:

$$P_{2,\mu}^\varsigma = -h^2 U_\mu \Delta_x U_\mu^{-1} + \lambda_2(x + \mu v(x)) \quad (8)$$

at the neighbourhood of point x_0 of the well such that $(\forall \varepsilon' \rangle 0$,small enough, $||x_0|\rangle r_0 + \varepsilon'$), the distorsion $P_{2,\mu}$ is in fact a dilatation of angle θ such that $e^\theta = (1+\mu)$. We denote it by $P_{2,\theta}$ [11] and is defined by

$$P_{2,\theta} = -h^2 \Delta_x + \lambda_2(xe^\theta) \quad (9)$$

Let $e_j, j = 1, ..., N_0$ be the eigenvalues of the operator $P_0 = -\dfrac{d^2}{dr^2} + \dfrac{1}{2}\lambda_2^{"}(r_0)(r-r_0)^2)$ and γ_j complex circles centred at e_j h.

Theorem 1.7: Under assumptions (H1)- (H5), for $\theta \in C, |\theta|$ and *h* small enough and for $(\forall \varepsilon' \rangle 0$,small enough, $||x_0|\rangle r_0 + \varepsilon'$), the resolvent of the distorted operator defined by (9) satisfies the estimate $\left\| \left(P_{2,\theta} - z\right)^{-1} \right\| = O\left(h^{-1/2}\right)$, uniformly for $z \in [-\varepsilon' - x_0, C_0 h - x_0]$ outside of the γ_j.

Before we prove this theorem, we introduce the so-called Grushin problem associated to the distorted operator P_μ.

The reduced Feshbach operator: Now, we try to reduce the operator P_μ^ς by the Feshbach method into a matricial operator of type: $-\dfrac{h^2}{(1+\mu)^2}\Delta_x I + M_\mu^\varsigma + \tilde{R}_\mu^\varsigma$

where M_μ^ς is the matrix of eigenvalues of Q_μ^ς and $\tilde{R}_\mu^\varsigma$ is the remainder of order $O(h^2)$

The study of the distorted operator P_μ^ς: We begin our study by the operator Q_μ^ς which is defined by:

$$Q_\mu^\varsigma = U_\mu Q(x + \mu v(x)) U_\mu^{-1} \quad (10)$$

For x ≠ 0, we denote also

$\tilde{Q}_\mu(x) = Q_\mu(x) - \dfrac{\alpha}{|x + \mu v(x)|}$ and $\tilde{\lambda}_j(x) = \lambda_j - \dfrac{\alpha}{|x|},\ j \in \{1,2,3\}$

Let C (x) be a family of continuous closed simple loop of C enclosing $\tilde{\lambda}_j(x),\ j \in \{1,2,3\}$ and having the rest of $\sigma(\tilde{Q}_0(x))$ in its exterior. The gap condition (4) permits us to assume that:

$$\min_{x \in IR^3} \operatorname{dist}(\gamma(x), \sigma(\tilde{Q}_0(x))) \geq \frac{\delta}{2} \quad (11)$$

Using the relation (6) and (H3), we can take C (x) compact in a set of C. So, we deduce from (11) the following result[3].

Lemma 2.1

1. $\forall j, k \in \{1, ..., p\}$, $j \neq k$, $\beta \in IN^{3p}$, the operators $\dfrac{1}{|y_j \pm x|}\left(\tilde{Q}_0(x) - z\right)^{-1}$, $\dfrac{1}{|y_j - y_k|}\left(\tilde{Q}_0(x) - z\right)^{-1}$ and $\partial^\beta\left(\tilde{Q}_0(x) - z\right)^{-1}$ are uniformly bounded on $L^2(IR_y^{3p})$, $x \in IR^3$, $z \in C(x)$

2. If $\mu \in$ small enough, then for $x \in IR^3$, $z \in$,the operator $\left(\tilde{Q}_\mu(x) - z\right)^{-1}$ exists and satisfies uniformly $\left(\tilde{Q}_\mu(x) - z\right)^{-1} - \left(\tilde{Q}_0(x) - z\right)^{-1} = O|\mu|$.

Now we define for $\mu \in C$ small enough, the spectral projector associated to $\tilde{Q}_\mu$ and the interior of $C(x)$.

$$\pi_\mu(x) = \frac{1}{2\pi}\int_{\gamma(x)} (z - \tilde{Q}_\mu(x))^{-1} \text{ and } rg\pi_\mu = 1$$

This projector permits us to construct the Grushin problem associated to the operator P_μ^ς.

Problem of Grushin associated with the operator P_μ^ς: We begin this section by the result which is (lemma1-1 of[12] and proposition 5-1 of[7].

Proposition 2.2: Assume (H1), (1.7), (1.9), (1,10) hold, then for $\mu \in C$, $z \in C$ small enough, there exist N functions $\omega_{k,\mu}(x,y) \in C^0(IR^3, H^2(IR^{3p}))$, (k = 1,2,3), depending analytically on $\mu \in$, such that

i. $\langle \omega_{j,\mu} | \omega_{k,\mu} \rangle_{L^{\acute{e}}(IR^{3p})} = \delta_{j,k}$

ii. For $|x| \geq \frac{3}{N}, (\omega_{k,\mu})_{1\leq k\leq 3}$ form a basis of $\operatorname{Ran}\pi_\mu(x)$

iii. $\in C^\infty\left(\left\{|x|\langle\frac{2}{N}\right\}, H^2(IR^{3p})\right)$

iv. For $|x|$ large enough, $\omega_{k,\mu}(x)(x)$ is an eigen function of $Q_\mu(x)$ associated with $\lambda_k(x + \mu\omega(x))$

We first introduce the family $\{\omega_{1,\mu}, \omega_{2,\mu}, \omega_{3,\mu}\}$ of $\operatorname{Ran}\pi_\mu(x)$ depending analytically on μ for μ small enough and normalized in $L^2(IR_y^{3p})$ by $\langle \omega_{i,\mu}(x), \omega_{j,\bar{\mu}}(x)\rangle_{L^2(IR_y^{3p})} = \delta_{ij}$ and then we associate the two following operators

$$R_\mu^- : \bigoplus_1^3 L^2(IR^3) \to L^2(IR^{3p})$$

$$u^- = (u_1^-, u_2^-, u_3^-) \to R_\mu^- u^- = \sum_{k=1}^3 u_k^- \omega_{k,\mu}(x)$$

$$R_\mu^+ = (R_\mu^-)^* : L^2(IR^{3p}) \to \bigoplus_1^3 L^2(IR^3)$$

$$u = {}^t(\langle u, \omega_{\bar{\mu},1}\rangle_Y, \langle u, \omega_{\bar{\mu},2}\rangle_Y, \langle u, \omega_{\bar{\mu},3}\rangle_Y$$

where ${}^tA$ denote the transposed of the operator A, $\langle .,.\rangle_Y$ the inner product on $L^2(IR^{3p})$ and $\langle ., \omega_{\bar{\mu},1}\rangle_Y$ is the adjoin of the operator $L^2(IR^n) \ni v \mapsto vu_{\mu,j} \in L^2(IR^{n+P})$,

$u_{\mu,k} = u(x + \mu v(x))$ and we put $\hat{\pi}_\mu = 1 - \pi_\mu$, where $\pi_\mu = \langle u, \omega_{\bar{\mu},1}\rangle_Y \omega_{\mu,1} + \langle u, \omega_{\bar{\mu},1}\rangle_Y \omega_{\mu,2} + \langle u, \omega_{\bar{\mu},3}\rangle_Y \omega_{\mu,3}$.

As P_μ^ς and $\omega_{\mu,k}$, $k = 1,2,3$ have analytic extensions with μ, the Grushin problem is then defined, for $z \in C$, by:

$$P_\mu^\varsigma(z) = \begin{pmatrix} P_\mu^\varsigma - z & R_\mu^+ \\ R_\mu^- & 0 \end{pmatrix} = \begin{pmatrix} P_\mu^\varsigma - z & \omega_{1,\mu} & \omega_{2,\mu} & \omega_{3,\mu} \\ \langle ., \omega_{1,\mu}\rangle_Y & 0 & 0 & 0 \\ \langle ., \omega_{2,\mu}\rangle_Y & 0 & 0 & 0 \\ \langle ., \omega_{3,\mu}\rangle_Y & 0 & 0 & 0 \end{pmatrix} \qquad (12)$$

which sets on $H^2(IR^{3p}) \oplus (\bigoplus_1^3 L^2(IR^3))$ to $L^2(IR^{3p}) \oplus (\bigoplus_1^3 H^2(IR^3))$

The following proposition, gives the inverse of the operator (12) by using a result of Grushin problem. This is proved in[3,6].

Proposition 2.3: $\forall z \in C$ close enough to λ_0, P_μ^ς is invertible and we can write its inverse:

$$P_\mu^{\varsigma-1} = \begin{pmatrix} X_\mu^\varsigma & X_{\mu,+}^\varsigma \\ X_{\mu,-}^\varsigma & X_{\mu,-+}^\varsigma \end{pmatrix},$$

With $X_\mu^\varsigma(z) = (P'^\varsigma_\mu - z)^{-1}\hat{\pi}_\mu(x)$ where $(P'^\varsigma_\mu - z)^{-1}$ is the bounded inverse of the restriction of $\hat{\pi}_\mu(P_\mu^\varsigma - z)$ to $\{u \in H^2(IR^{3(n+p)}, \hat{\pi}u = u\}$.

$$X_{\mu,+}^\varsigma(z) = (\omega_{k,\mu} - X_\mu^\varsigma(z)P_\mu^\varsigma(.\omega_{k,\mu}))_{1\leq k\leq 3},$$

$$X_{\mu,-}^\varsigma(z) = {}^t\left(\langle (1 - P_\mu^\varsigma(z)X_\mu^\varsigma)(.), \omega_{k,\bar{\mu}}\rangle_{1\leq k\leq 3}\right) \text{ and}$$

$$X_{\mu,-+}^\varsigma(z) = \left(z\delta_{jk} - \left\langle \left(P_\mu^\varsigma - P_\mu^\varsigma X_\mu^\varsigma(x)P_\mu^\varsigma\right)(.\omega_{j,\mu}), \omega_{j,\bar{\mu}}\right\rangle_{L^2(IR^{3p})}\right)_{1\leq j,k\leq 3}$$

Remark 2.4

1. For $z \in C$, close enough to λ_0, we have $z \in \sigma(P_\mu^\varsigma)$ if and only if $\exists\mu, |\mu|$ small enough and $\operatorname{Im}\mu\rangle 0$, such that $z \in \sigma_{disc}(X_{\mu,-+}^\varsigma(z))$ where $X_{\mu,-+}^\varsigma(z) : \bigoplus_1^3 H^2(IR^3) \to L^2(IR^3)$, is a pseudo-differential operator of principal symbol defined by the matrix:

$$B(x,\xi,z) = zI - (\langle \omega_{j,\mu}(x) | (t_\mu(\xi) + Q_\mu^\varsigma(x))\omega_{k,\mu}(x)\rangle_{L^2(IR^3)})_{1\leq j,k\leq 3}$$

and $t_\mu(\xi)$ is the principal symbol of $-h^2U_\mu\Delta_x U_\mu^{-1}$

2. z is a resonance of the operator P_μ^ς only and only if, $\exists\mu \in C$, $|\mu|$ small enough $\operatorname{Im}\mu\rangle 0$, such that: $0 \in \sigma_{disc}(X_{\mu,-+})$ or $0 \in \sigma_{disc}(F_{\mu\mu,-+}^\varsigma)$ where F_μ^ς is the Feshbach operator ($F_\mu^\varsigma = z - X_{-+\mu}^\varsigma$) our goal is to takeback the initial problem to a problem on $L^2(IR^3) \oplus L^2(IR^3) \oplus L^2(IR^3)$.

Reduced Feshbach operator: To reduce the Feshbach operator in a matricial operator, we input:

$$\Phi_\mu^\varsigma = P_\mu^\varsigma - P_\mu^\varsigma X_\mu^\varsigma(x) P_\mu^\varsigma \quad (13)$$

$$F_\mu^\varsigma = (\langle \Phi_\mu^\varsigma(.\omega_{j,\mu}(x)) | \omega_{k,\bar{\mu}}(x) \rangle_Y)_{1\le j,k\le 3} \quad (14)$$

and

$$\Phi_{1,\mu}^\varsigma(z) = (\langle \Phi_\mu^\varsigma(.\omega_{1,\mu}(x)) | \omega_{1,\bar{\mu}}(x) \rangle\ )_{1\le j,k\le 3} \quad (15)$$

The following proposition give us the estimation of the resolvent of the operator (15).

Proposition 2.5: For $z \in C, |z|$ small enough, $\mu \in C, |\mu|$ small enough, the operate or $(\Phi_\mu^{1\varsigma}(z) - z)$ is bijective for $H^2(IR^3)$ to $L^2(IR^3)$. Its inverse is extended for H^m in H^{m+j}
$H^m = H^m(L^2(IR_x^n, L^2(IR^p), \forall m \in Z$ and verify for $j = \{1,2,3\}$, h>0 small enough:

$$\left\| (\Phi_{1,\mu}^\varsigma(z) - z)^{-1} \right\|_{L(H^m, H^{m+j})} \le \frac{C(m)}{h^j (\operatorname{Im}\mu)}$$

To prove this proposition, we first use a lemma in[3], to prove the following lemma:

Lemma 2.6: �$\forall m \in Z$, the operator $X_\mu^\varsigma(z)$ is uniformely is extensible in a bounded operator on $H^m(L^2(IR_x^n), L^2(IR^p)), \forall m \in\ Z$, for $h \rangle 0$, $z \in Z$ and $\mu \in Z$ small enough and

$$\left\| X_\mu^\varsigma \right\|_{L(H^m, H^{m+2})} = O(h^{-2})$$

See [3] for the proof.

Lemma 2.7: We assume that

$$\left\| (P_{1,\mu}^\varsigma - z)^{-1} \right\|_{L^2(H^m, H^{m+j})} = O\left(\frac{1}{h^j \operatorname{Im}\mu}\right)$$

for $h \rangle 0$, $z \in C$ and $\mu \in C$ small enough, where

$$P_{1,\mu}^\varsigma = -h^2 \frac{1}{(1+\mu)^2} \Delta_x + \lambda_1(x + \mu v(x)) -$$
$$h^2 \frac{1}{(1+\mu)^2} \langle \Delta_x(.\omega_{1,\mu}(x) | \omega_{1,\bar{\mu}}(x) \rangle_Y -$$
$$-h^2 \langle R_\mu(x, D_x)(.\omega_{1,\mu}(x) | \omega_{1,\bar{\mu}}(x) \rangle_Y$$

$R_\mu(x, D_x)$, is an differiential operator of coefficients C^∞.

Proof of lemma 2.7: Using (H5) we have: $\operatorname{Im} \frac{1}{(1+\mu)^2} \lambda_1(x + \mu v(x)) \le -\frac{\operatorname{Im}\mu}{C_1}$, so

$$\left\| (-h^2 \frac{1}{(1+\mu)^2} \Delta_x + \lambda_1(x + \mu v(x)) - z)^{-1} \right\|_{L(L^2(IR^n))} \le \frac{C_2}{\operatorname{Im}\mu}$$

and we easily deduce with a simple computation that

$$\left\| (P_{1,\mu}^\varsigma - z)^{-1} \right\|_{L^2(H^m, H^{m+j})} = O\left(\frac{1}{h^j \operatorname{Im}\mu}\right)$$

Proof of the proposition 2.5: From (13) and (15), we have $\Phi_{1,\mu}^\varsigma = \langle (P_\mu^\varsigma - P_\mu^\varsigma X_\mu^\varsigma(z) P_\mu^\varsigma(.\omega_{1,\mu}(x) | \omega_{1,\bar{\mu}}(x) \rangle$, then we subtitue P_μ^ς from (7) with

$U_\mu \Delta_x U_\mu^{-1} = \frac{1}{(1+\mu)^2} \Delta_x + R_\mu(x, D_x)$, where $R_\mu(x, D_x)$ is a second order differential operator with C^∞ coefficients in x with compact support, analytic in μ and whose derivative of any kind compared to x are $O(|\mu|)$: and we put

$$\Lambda_\mu^\varsigma = \frac{1}{(1+\mu)^4} \langle \Delta_x X_\mu^\varsigma \Delta_x(.\omega_{1,\mu}(x)), \omega_{1,\bar{\mu}}(x) \rangle_Y +$$
$$+ \frac{1}{(1+\mu)^2} \left\langle \begin{matrix} (R_\mu(x, D_x) X_\mu^\varsigma \Delta_x + \Delta_x X_\mu^\varsigma R_\mu(x, D_x)) \\ (.\omega_{1,\mu}(x)), \omega_{1,\bar{\mu}}(x) \end{matrix} \right\rangle_Y .$$

Using the fact that

$\hat{\pi}_\mu \omega_{1,\mu} = 0,\ X_\mu^\varsigma = \hat{\pi}_\mu X_\mu^\varsigma \hat{\pi}_\mu$, $\langle \omega_{1,\mu}, \omega_{1,\bar{\mu}} \rangle = 1$, we have: $\Phi_{1,\mu}^\varsigma(z) = \breve{P}_{1,\mu}^\varsigma - h^4 \Lambda_\mu^\varsigma$, where

$$\breve{P}_{1,\mu}^\varsigma = -h^2 \frac{1}{(1+\mu)^2} \Delta_x + \lambda_1(x + \mu v(x))$$
$$-\frac{1}{(1+\mu)^2} \langle \Delta_x(.\omega_{1,\mu}(x) | \omega_{1,\bar{\mu}}(x) \rangle_Y$$
$$-h^2 \langle R_\mu(x, D_x)(.\omega_{1,\mu}(x) | \omega_{1,\bar{\mu}}(x) \rangle_Y$$

We have $R_x(x, D_x)$ bounded, so Λ_μ^ς is $O(h^2)$ from H^m to H^m and we also see from (H5) and lemma2.6 that: for h small enough, $\left\| (P_{1,\mu}^\varsigma - z)^{-1} \right\|_{L(L^2)} = O\left(\frac{1}{\operatorname{Im}\mu}\right)$, then, we deduce

$\left\| (\breve{P}_{1,\mu}^\varsigma - z)^{-1} \right\|_{L^2(H^m, H^{m+j})} = O\left(\frac{1}{h^j \operatorname{Im}\mu}\right)$. Finally we have:

$$\left\| (\Phi_{1,\mu}^\varsigma(z) - z)^{-1} \right\|_{L(H^m, H^{m+j})} = O\left(\frac{1}{h^j \operatorname{Im}\mu}\right)$$

Proof of theorems

Proof of theorem 2.1: Proposition3.5 permits us to reduce the Feshbach operator F_μ^ς in a matricial operator

2x2, A_μ^ς, where

$A_\mu^\varsigma = \left\{ \left\langle \Phi_\mu^\varsigma(.\omega_{i,\mu}) + T_\mu^{j\varsigma}(.\omega_{1,\mu}), \omega_{1,\bar{\mu}} \right\rangle_Y \right\}_{i,j=2,3}$

Now, we consider a solution $\alpha = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \in L^2(IR^n) \oplus L^2(IR^n) \oplus L^2(IR^n)$ of the equation: $F_\mu^\varsigma(z)\alpha = z\alpha$

The operators $T_\mu^{j\varsigma}$ are defined by:

$T_\mu^{j\varsigma}(z)\alpha_j = -(\Phi_\mu^{1\varsigma}(z) - z)^{-1} \left\{ \left\langle \Phi_\mu^\varsigma(\alpha_j \omega_{j,\mu}, \omega_{j,\bar{\mu}} \right\rangle_Y \right\}_{j=2,3}$,

hence, the spectral study of the Feshbach F_μ^ς becomes the study of the operator A_μ^ς on $L^2(IR^n) \oplus L^2(IR^n)$ by:

$\alpha_1 = -(\Phi_\mu^{1\varsigma}(z) - z)^{-1} = \left\{ \left\langle \Phi_\mu^\varsigma(\alpha_2 \omega_{2,\mu}, \omega_{2,\bar{\mu}} \right\rangle_Y + \left\langle \Phi_\mu^\varsigma(\alpha_2 \omega_{3,\mu}, \omega_{3,\bar{\mu}} \right\rangle_Y \right\}$

Then the eigenvalues equation of $F_\mu^\varsigma(z)$ becomes:

$$\begin{cases} \alpha_1 = (T_\mu^{2\varsigma}(z) \oplus T_\mu^{3\varsigma}(z))(\alpha_2 \oplus \alpha_3) \\ A_\mu^\varsigma(z)(\alpha_2 \oplus \alpha_3) = z(\alpha_2 \oplus \alpha_3) \end{cases}$$

So we establish easily

$A_\mu^\varsigma = -h^2 \dfrac{1}{(1+\mu)^2} \Delta_x + M_\mu^\varsigma + \tilde{R}_\mu^\varsigma$, where M_μ^ς is a diagonal matrix outside of $K_{2\delta_0}$ and it equal to:

$$M_\mu^\varsigma = \left\{ \left\langle Q_\mu^\varsigma(x)(.\omega_{i,\mu}) \middle| \omega_{j,\bar{\mu}} \right\rangle_Y \right\}_{i,j=2,3} = \begin{pmatrix} \lambda_2(x+\mu v(x)) & 0 \\ 0 & \lambda_3(x+\mu v(x)) \end{pmatrix}$$

where $\lambda_2(x+\mu v(x))$, $\lambda_3(x+\mu v(x))$ are the eigenvalues of Q_μ^ς, $\forall x \in IR - \{0\}$

The remainder

$\left\| \tilde{R}_\mu^\varsigma(z,h) \right\|_{L(H^m \oplus H^m, H^{m-1} \oplus H^{m-1}} = O(h^2)$, $\forall m \in Z$ uniformly for $h \rangle 0$ and $z \in C$ closed to λ_0

At the end we prove the second result. To describe it, we apply a technical of Briet Combs Duclos[13].

Let $J_i \in C_0^\infty(|x - x_0| \le \delta), (\delta \rangle 0$ fixed small enough and x_0 a point of maximum) and $J_e \in C^\infty(IR^n)$ such that: $J_i = 1$ near x_0 and $J_i^2 + J_e^2 = 1$

J is an identification mapping such that:

$$J : L^2(IR^n) \oplus L^2(\operatorname{supp} J_e) \to L^2(IR^n)$$
$$J(u \oplus w) = J_i u + J_e w$$

It is easily proved that: $JJ^* = 1_{L^2(IR^n)}$

Now, if we note P_μ^Ω the Dirichlet realisation of P_μ^ς on Ω, on Ω, $x = v(x)$ and the distorsion $x + \mu v(x) = xe^\theta$, is an analytic dilatation (whose Dirichlet realisation is the operator H_μ^ς obtained for $\varsigma = 1$)). We set

$H_\theta^i = -h^2 e^{-2\theta}\Delta + \left\langle \lambda_2''(x_0)(x - x_0), (x - x_0) \right\rangle e^{2\theta}$

$H_\theta = P_\theta^2 = -h^2 e^{-2\theta}\Delta + \lambda_2(xe^\theta)$

$H_\theta^e = H_\theta \Big|_{L^2(\operatorname{supp} J_e)}$, with Dirichlet conditions on $\partial \operatorname{supp} J_e$

Remark 3.1: Since $\inf_{x \in \operatorname{supp} J_e} \operatorname{Re} e^{2\theta}\lambda_2(xe^\theta) \rangle 0$, $(H_\theta^e - z)^{-1}$ is uniformly bounded for $|z|$ and *h* small enough.

Before we prove the second result, we introduce the following lemma

Lemma 3.2: For all $p \in [0,1]$, $\left\| |x|^p (H_\theta^i - z)^{-1} \right\|_{L(L^2)} = O(h^{\frac{p}{2}-\frac{1}{2}})$, uniformly for z outside of $\gamma(x)$ $z \in [-\varepsilon - x_0, C_0 h - x_0] + i[-\varepsilon - x_0, C_0 h - x_0]$, $\operatorname{Im}\theta \ge 0$, and *h* small enough.

Proof of lemma 3.2: If we put $y = \dfrac{x - x_0}{\sqrt{h}}$, we can write H_i^θ:

$$H_i^\theta = hH_i^0 \tag{16}$$

where $H_i^0 = -e^{-2\theta}\Delta_y + \dfrac{1}{2}\left\langle \lambda''(x_0)y, y \right\rangle + h^{-1}\Im(\varepsilon)$,

with $\Im(\varepsilon) = \varepsilon(1 + (x - x_0)e^\theta + \dfrac{1}{2}(x - x_0)^2 e^{2\theta})$

It is enough to show that, for $\theta = i\alpha, \alpha \ge 0$, small enough. We have from (16)

$$|x|^p (H_\theta^i - z)^{-1} = h^{\frac{p}{2}-\frac{1}{2}} |y|^2 (H_\theta^0 - zh^{-1})^{-1} \tag{17}$$

and the eigenvalues of the operator H_i^0 in

$]-\infty, C_0 - x_0] + i\,IR$ are $e_1, ..., e_N$.

We distinguish three cases for $p = 0$.

1/ If $z \in [-Ch - x_0, C_0 h - x_0] + i[-Ch - x_0, C_0 h - x_0]$: we deduce for all C□0, $(H_\theta^0 - zh^{-1})^{-1}$ is bounded on L² uniformly for z outside the γ_j, so (17) is verified.

2/ If $z \in [-\varepsilon - x_0, C_0 h - x_0] + i[-\varepsilon - x_0, Ch - x_0]$: then for $u \in C_0^\infty(IR^n)$:

$$e^{2\theta}H_\theta^0 = -\Delta y + \frac{1}{2}\langle\lambda''(x_0)y, y\rangle e^{4\theta} +$$

$$h^{-1}(z + \varepsilon(1 + (x - x_0)e^{3\theta} + \frac{1}{2}(x - x_0)^2 e^{4\theta})$$

and

$$\operatorname{Im}\left\langle e^{2\theta}(H_\theta^0 - zh^{-1})u, u\right\rangle = \frac{1}{2}\sin 4\alpha\left\langle\langle\lambda''(x_0)y, y\rangle u, u\right\rangle -$$

$$-\left[h^{-1}(z\sin 2\alpha + \operatorname{Im} z\cos 2\alpha + h^{-\frac{1}{2}}(y\sin 3\alpha + z\cos 4\alpha)\right]\|u\|^2$$

We take particularly α small enough and *C* large enough such that: $C\cos 2\alpha \rangle C_0 \sin 2\alpha$

At least we obtained

$$\left|\left\langle e^{2\theta}(H_\theta^0 - zh^{-1})u, u\right\rangle\right| \geq h^{-\frac{1}{2}}(x_0 \sin 2\alpha + y\sin 3\alpha)\|u\|^2 \text{ so}$$

the result is also verified. It remain the case:

3/ If $z \in [-\varepsilon - x_0, -Ch - x_0] + i[-Ch - x_0, C_0 h - x_0]$:

$$\operatorname{Re}\left\langle e^{2\theta}(H_\theta^0 - zh^{-1})u, u\right\rangle$$

$$\geq h^{-\frac{1}{2}}(\operatorname{Re} z\cos 4\alpha - \operatorname{Im} z\sin 2\alpha + y\cos 3\alpha),$$

we deduce the estimation when $C \rangle C_0$, α small enough and *C* large enough such that $\cos 4\alpha \rangle \sin 2\alpha$

Now we consider the case when $p \neq 0$,

$$e^{2\theta}(H_\theta^0 - zh^{-1}) = -\Delta + \frac{1}{2}e^{4\theta}\langle\lambda''(x_0)y, y\rangle \text{ and}$$

$$-zh^{-1}e^{2\theta} + h^{-1}e^{2\theta}\Im(\varepsilon)$$

$$\left\|-\Delta + \frac{1}{2}e^{4\theta}\langle\lambda''(x_0)y, y\rangle - zh^{-1}e^{2\theta} + h^{-1}e^{2\theta}\Im(\varepsilon)\right\|$$

$$\geq \left\|\frac{1}{2}\cos 4\alpha\langle\lambda''(x_0)y, y\rangle u\right\|_{L^2} \geq \frac{1}{C}|y|^2\|u\|_{L^2}$$

if we put $u = (H_\theta^0 - zh^{-1})^{-1}v$ the result is deduced from a priori standard estimation.

Proof of theorem 1.2: We put $H_\theta^d = H_\theta^i \oplus H_\theta^e$ and $\Pi = H_\theta J - JH_\theta^d$, for z outside the spectrum of H_θ, with a simple calculation we obtain:

$$(H_\theta - z)^{-1} = J(H_\theta^d - z)^{-1}J^*(1 + \Pi(H_\theta^d - z)^{-1}J^*)^{-1} \quad (18)$$

Using the lemma3.2 (with $p = 2$) and the lemma3.1of Briet Combs Duclos[13], we can easily prove that: $\exists\beta\langle 1$ such that

$$\left\|\Pi(H_\theta^d - z)^{-1}J^*\right\| \leq \beta \quad (19)$$

Using the lemma3.2 and (19),we obtain from (18)

$\left\|(H_\theta - z)^{-1}\right\| \leq C\left\|(H_\theta^d - z)^{-1}\right\|$, finally the result is obtained from lemma3.2 and remark3.1

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