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On Some Stability Results for Fixed Point Iteration Procedure

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Abstract: In this study, we establish that both the Mann and Ishikawa iteration processes are T-stable for the mappings T satisfying a more general contractive definition than that of Osilike[1] . The results obtained generalize some of the recent results of Osilike[1] which are themselves generalizations and extensions of some of the results of Harder and Hicks[2] and Rhoades[3,4].

Key words: Stability results, fixed point iteration procedure

INTRODUCTION

Let (E, d) be a complete metric space and $T : E \to E$ a selfmap of E and $F(T) = \{p \in E : T_p = p\}$, the set of fixed points of T. For $x_o \in E$, define sequence $\{x_x\}_{n=o}^{\infty}$ iteratively by

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots \tag{1}$$

Suppose $\{x_n\}_{n=o}^{\infty}$ converges to a fixed point p of T and let $\varepsilon_n = d(y_{n+1}, f(T, y_n))$, where $\{y_n\}_{n=o}^{\infty} \subset E$. Then, the iteration procedure (I) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

Harder and Hicks[2] established several stability results under various contractive conditions using the above concept. Rhoades[3,4] extended the results of Harder and Hicks[2] to other classes of contractive mappings. In Rhoades[4], the following contractive definition was considered : there exists a constant $c \in [0,1)$, such that for each $x, y \in E$,

$$d(Tx, Ty) \le c \max\{d(x, y), \tfrac{1}{2}[d(x, Tx) + d(y, Ty)], d(x, Ty)], d(x, Ty), d(y, Tx)\} \tag{2}$$

Using (2), Rhoades[4] established several stability results which are generalizations and extensions of most of the results of Harder and Hicks[2] and Rhoades[5]. It was shown in Rhoades[6] that if T satisfies (2) then,

$$d(Tx, Ty) \le \frac{c}{1-c} d(x, Ty) + c\, d(x, y)$$

Osilike[1] employed the following contractive definition: for each $x, y \in E$, there exist constants $a \in [0,1)$ and $L \ge 0$ such that

$$d(Tx, Ty) \le L d(x, Tx) + a\, d(x, y). \tag{3}$$

Using (3), Osilike[1] proved several stability results which are generalizations and extensions of most of the results of Rhoades[4].

Employing the same contractive definitions as in Harder and Hicks[2], Berinde[7] proved the same stability results for the same iteration procedures by an alternative method.

In this study, we extend some of the recent results of Berinde[7], Osilike[1] and Rhoades[4] to a more general contractive definition.

Preliminaries: In the sequel, we shall employ the following contractive definition. For each $x, y \in E$, there exist a constant $b \in [0,1\}$, and a continuous, monotone increasing function $\varphi : \Re_+ \to \Re_+$ with $\varphi(0) = 0$, such that

$$d(Tx, Ty) \le \varphi(d(x, Tx)) + b d(x, y). \tag{4}$$

The contractive definition (4) is more general than those considered by Berinde[7], Harder and Hicks[2], Rhoades[3,4] and Osilike[1]. This is evident by specifying φ in (4) as follows. If $\varphi(u) = Lu$ in (4) above, where $L \ge 0$ is a constant, then we obtain the contractive mapping of Osilike[1] which is itself a generalization of those in Harder and Hicks[2], Berinde[7] and Rhoades[4]. Also, if $L = mb$, where $m = (1-b)^{-1}$, $b \in [0,1)$, we obtain the contractive mapping considered by Rhoades[4].

Also, if $L = 2\delta$, $b = \delta$, where $\delta = \max\left\{\alpha \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $0 \le \alpha < 1$, $0 \le \beta < 0.5$, $0 \le \Upsilon \le 0.5$, then we obtain the Zamfirescu's contractive definition which was employed in Harder and Hicks[2] and Berinde[7]. Furthermore, if $\varphi(u) = 0$, then (4) reduces to $d(Tx, Ty) \le bd(x, y)$, $b \in [0,1)$ which is another contractive definition used by Harder and Hicks[2] and Berinde[7].

In the sequel, we shall establish stability results for the following iteration procedures:

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i. The Mann Iteration Process [1,6], which is defined for arbitrary $x_o \in E$ by $x_{n+1} = f(T, x_n) = (1-\alpha_n)x_n + \alpha_n Tx_n$, $n \geq 0$, Where, $\{\alpha_n\}_{n=o}^{\infty}$ is a real sequence satisfying $\alpha_o = 1$, $0 \leq \alpha_n \leq 1$, for $n > 0$ and $\sum_{n=o}^{\infty}\alpha_n = \infty$.

ii. The Ishikawa Iteration Process [1,6] which is defined for arbitrary $x_o \in E$ by:

$$\left.\begin{aligned} Z_n &= (1-\beta_n)x_n + \alpha_n Tx_n \\ x_{n+1} &= f(T, X_n) = (1-\alpha_n)x_n + \alpha_n Tx_n, \end{aligned}\right\}$$

Where, $\{\alpha_n\}_{n=o}^{\infty}$ and $\{\beta_n\}_{n=o}^{\infty}$ are real sequences satisfying $0 \leq \alpha_n \leq \beta_n \leq 1$ for all $n \geq 0$, and $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=o}^{\infty}\alpha_n\beta_n = \infty$. We shall employ the following lemmas in the proofs of the stability results.

Lemma 1: Let (E, d) be a complete metric space, and $T: E \to E$ - a selfmap of E satisfying (4). Let $x_o \in E$ and $x_{n+1} = Tx_n$, $n \geq 0$. Suppose T has a fixed point p and $\varphi : \Re_+ \to \Re_+ = [0, \infty)$ is a continuous monotone increasing function such that $\varphi(0) = 0$. Then, $\lim_{n=\infty}\varphi(d(x_n, Tx_n)) = 0$.

Proof: From (4) and the hypothesis of the Lemma, we have:

$$\begin{aligned} d(x_{n+1}, p) &= d(Tx_n, Tp) = d(Tp, Tx_n) \\ &\leq \varphi(d(p, Tp)) + bd(p, x_n) \\ &= bd(x_n, p) \leq b^2 d(x_{n-1}, p) \leq \ldots \\ &\leq b^{n+1}d(x_o, p) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

By triangle inequality and (4), we have:

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, p) + d(p, Tx_n) \\ &= d(x_n, p) + d(Tp, Tx_n) \\ &\leq d(x_n, p) + \varphi(d(p, Tp)) + bd(p, x_n) \\ &= (1+b)d(x_n, p) \to 0 \text{ as } n \to \infty \end{aligned}$$

Thus, $\lim_{n=\infty} d(x_n, Tx_n) = 0$.

But φ is continuous, therefore we have :
$\lim_{n\to\infty}\varphi(d(x_n, Tx_n)) = \varphi(\lim_{n\to\infty} d(x_n, Tx_n)) = 0$,
This completes the proof of the Lemma.

Remark 1: The operator T in Lemma 1 is not necessarily a Picard operator.

Lemma 2: Let $(E, \|.\|)$ be a normed linear space, and let $T : E \to E$ be a selfmap of E satisfying (4). Suppose T has a fixed point p. Let $\{x_n\}_{n=o}^{\infty}$ be the Ishikawa iteration process with $\{\alpha_n\}_{n=o}^{\infty}$ and $\{\beta_n\}_{n=o}^{\infty}$ satisfying

i. $\alpha_o = 1$;
ii. $0 \leq \alpha_n$, $\beta_n \leq 1$, $n \geq 0$;
iii. $\sum_{n=o}^{\infty}\alpha_j = \infty$;
iv. $\sum_{n=o}^{\infty}\prod_{k=j+1}^{n}(1-\alpha_k + b\alpha_k)$ converges.

Suppose $\varphi : \Re_+ \to \Re_+$ is a continuous monotone increasing function such that $\varphi(0) = 0$. Let $\{y_n\}_{n=o}^{\infty} \subset E$ and define.

$s_n = (1-\beta_n)y_n + \beta_n Ty_n$, $n \geq 0$
$\varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\|$

Then,

$$\begin{aligned} \|y_{n+1} - p\| \leq \|x_{n+1} - p\| &+ \sum_{j=o}^{\infty}\prod_{k=j+1}^{n}(1-\alpha_k + b\alpha_k)\,\varphi\|z_j - Tz_j\|) \\ &+ \sum_{j=o}^{\infty}\alpha_j\beta_j\prod_{k=j+1}^{n}(1-\alpha_k + b\alpha_k)\,\varphi(\|x_j - Tx_j\|) \\ &+ \prod_{k=o}^{n}(1-\alpha_k + b\alpha_k)\,\|x_o - y_o\| \\ &+ \sum_{j=o}^{\infty}\prod_{k=j+1}^{n}(1-\alpha_k + b\alpha_k)\varepsilon_j, \end{aligned} \tag{5}$$

Where the product is 1 when j = n.

Proof: Using (4) and the triangle inequality, we have the following:

$$\begin{aligned} \|y_{n+1} - p\| &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - p\| \\ &\leq \|y_{n+1} - p\| + \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\| \\ &\quad + \|(1-\alpha_n y_n - \alpha_n Ts_n - x_{n+1}\| \\ &= \|x_{n+1} - p\| + \varepsilon_j + \| \\ &\quad (1-\alpha_n)y_n - \alpha_n Ts_n - (1-\alpha_n)x_n - \alpha_n Tz_n\| \\ &\leq \|x_{n+1} - p\| + (1-\alpha_n) + \|x_n - y_n\| \\ &\quad + \alpha_n\|Tz_n - Ts_n\| + \varepsilon_n \\ &\leq \|x_{n+1} - p\| + (1-\alpha_n)\|x_n - y_n\| \\ &\quad + \alpha_n[\varphi(\|z_n - Tz_n\|) + b\|z_n - s_n\|] + \varepsilon_n \\ &= \|x_{n+1} - p\| + (1-\alpha_n)\|x_n - y_n\| \\ &\quad + \alpha_n\varphi(\|z_n - Tz_n\|) + \alpha_n b\|z_n - s_n\| + \varepsilon_n \end{aligned} \tag{6}$$

Observe that

$$\begin{aligned} \|z_n - s_n\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - (1-\beta_n)y_n - \beta_n Ty_n\| \\ &\leq (1-\beta_n)\|x_n - y_n\| + \beta_n\|Tx_n - Ty_n\| \\ &\leq (1-\beta_n)\|x_n - y_n\| + \beta_n(\varphi\|x_n - Tx_n\|) \\ &\quad + b\|x_n - y_n\|) \\ &= \beta_n\varphi(\|x_n - Tx_n\|) + (1-\beta_n + b\beta_n)\|x_n - y_n\| \\ &\leq \beta_n\varphi(\|x_n - Tx_n\|) + \|x_n - y_n\|. \end{aligned} \tag{7}$$

Substituting (7) into (6), we have

$$\begin{aligned} \|y_{n+1} - p\| &\leq \|y_{n+1} - p\| + \alpha_n\varphi \\ &(\|z_n - Tz_n\|) + b\alpha_n\beta_n\varphi(\|x_n - Tx_n\|) \\ &+ (1-\alpha_n + b\alpha_n)\|x_n - y_n\| + \varepsilon_n \end{aligned} \tag{8}$$

Moreover,

$$\begin{aligned} \|y_n - x_n\| &\leq \|y_n - (1-\alpha_{n-1})_{yn-1} - \alpha_{n-1}Ts_n\| \\ &\quad + \|(1-\alpha_{n-1})y_{n-1} + \alpha_{n-1}Ts_{n-1} - x_n\| \\ &= \varepsilon_{n-1} + \|(1-\alpha_{n-1})y_{n-1} - \alpha_{n-1}Ts_{n-1} \\ &\quad - (1-\alpha_{n-1})x_{n-1} - \alpha_{n-1}Tz_{n-1}\| \end{aligned}$$

$\leq (1 - \alpha_{n-1}) \| x_{n-1} - y_{n-1} \| + \alpha_{n-1} \| Tz_{n-1} - Ts_{n-1} \| + \varepsilon_{n-1}$
$\leq (1 - \alpha_{n-1}) \| x_{n-1} - y_{n-1} \| + \alpha_{n-1} (\varphi(\| z_{n-1} - Tz_{n-1} \|) + b \| z_{n-1} - s_{n-1} \|) + \varepsilon_{n-1}$
$= \alpha_{n-1}\varphi(\| z_{n-1} - Tz_{n-1} \|) + (1 - \alpha_{n-1}) \| x_{n-1} - y_{n-1} \| + b\alpha_{n-1} \| z_{n-1} - s_{n-1} \| + \varepsilon_{n-1}$ (9)

Similarly, from (7), we have:

$$\| z_{n-1} - s_{n-1} \| \leq \beta_{n-1}\varphi(\| x_{n-1} - Tx_{n-1} \|) + \| x_{n-1} - y_{n-1} \| \quad (10)$$

Substituting (10) into (9), we have

$$\| x_n - y_n \| \leq \alpha_{n-1}\varphi(\| z_{n-1} - Tz_{n-1} \|) + b\alpha_{n-1}\beta_{n-1}\varphi(\| x_{n-1} - Tx_{n-1} \|) + (1 - \alpha_{n-1} + b\alpha_{n-1}) \| x_{n-1} - y_{n-1} \| + \varepsilon_{n-1} \quad (11)$$

Substituting (11) into (8) yields:

$\| y_{n+1} - p \| \leq \| x_{n+1} - p \| + \alpha_n\varphi(\| z_n - Tz_n \|) + b\alpha_n\beta_n\varphi(\| x_n - Tx_n \|) + \varepsilon$
$+ (1 - \alpha_n + b\alpha_n)\varepsilon_{n-1} + (1 - \alpha_n + b\alpha_n)\alpha_{n-1}\varphi(\| z_{n-1} - Tz_{n-1} \|)$
$+ b(1 - \alpha_n + b\alpha_n)\alpha_{n-1}\beta_{n-1}\varphi(\| x_{n-1} - Tx_{n-1} \|)$
$+ (1 - \alpha_n + b\alpha_n)(1 - \alpha_{n-1} + b\alpha_{n-1}) \| x_{n-1} - y_{n-1} \|$

Repeating this process (n-1) more times yields (5). This completes the proof.

Remark 2: If $\beta_n = 0$ in Lemma 2, then we obtain an equivalent result for the Mann iteration process.

MAIN RESULTS

Theorem 1: Let $(E, \|.\|)$ be a normed linear space and let $T : E \to E$ be a selfmap of E satisfying the contractive definition (4). Suppose T has a fixed point p and the sequence $\{x_n\}_{n=o}^{\infty}$ is the Ishikawa iteration process satisfying the conditions of Lemma 2. Then, the Ishikawa iteration process is T-stable.

Proof: Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. Then, we shall show that $\lim_{n\to\infty} y_n = p$, using Lemmas 1 and 2. Let C be the lower triangular matrix with entries:

$c_{nj} = \alpha_j \prod_{k=j+1o}^{n} (1 - \alpha_k + b\alpha_k)$. Then, C is multiplicative[1,4]. Since φ is continuous and $\lim_{n\to\infty} \| z_n - Tz_n \| = 0$, then by Lemma 1, we obtain:

$$\lim_{n\to\infty} \sum_{j=0}^{n} \alpha_j \prod_{k=j+1}^{n} (1 - \alpha_k + b\alpha_k)\varphi(\| z_j - Tz_j \|) = 0.$$

Furthermore,

$$0 \leq b\sum_{j=0}^{n} \alpha_j \prod_{k=j+1}^{n} (1 - \alpha_k + b\alpha_k)\varphi(\| x_j - Tx_j \|)$$
$$\leq b\sum_{j=0}^{n} \alpha_j \prod_{k=j+1}^{n} (1 - \alpha_k + b\alpha_k)\varphi(\| x_j - Tx_j \|).$$

Since φ is continuous and $\lim_{n\to\infty} \| x_n - Tx_n \|) = 0$, we have

$$\lim_{n\to\infty} b\sum_{j=0}^{n} \alpha_j \prod_{k=j+1}^{n} (1 - \alpha_k + b\alpha_k)\varphi(\| x_j - Tx_j \|) = 0.$$

which implies that :

$$\lim_{n\to\infty} b\sum_{j=0}^{n} \alpha_j \beta_j \prod_{k=j+1}^{n} (1 - \alpha_k + b\alpha_k)\varphi(\| x_j - Tx_j \|) = 0,$$

Let D be the lower triangular matrix with entries $d_{nj} = \prod_{k=j+1}^{n} (1 - \alpha_k + b\alpha_k)$.

Condition (iv) of Lemma 2 implies that D is multiplicative[5] and since $\lim_{n\to\infty} \varepsilon_n = 0$, we obtain:

$$\lim_{n\to\infty} \sum_{j=o}^{n} \prod_{k=j+1}^{n} (1 - \alpha_k + b\alpha_k)\varepsilon_n = 0$$

Moreover, condition (iii) of Lemma 2 implies that $\lim_{n\to\infty} \prod_{k=o}^{n} (1 - \alpha_k + b\alpha_k) = 0$.

Also, we shall prove that $\lim_{n\to\infty} \| x_{n+1} - p \| = 0$.

Using (4), triangle inequality and condition (ii) of Lemma 2, we have:

$$\begin{aligned}
\| x_{n+1} - p \| &= \| (1 - \alpha_n)x_n + \alpha_n Tz_n - p \| \\
&= \| (1 - \alpha_n)(x_n - p) + \alpha_n(Tz_n - p) \| \\
&= \| (1 - \alpha_n)(x_n - p) + \alpha_n(Tz_n - Tp) \| \\
&\leq \| (1 - \alpha_n) \| x_n - p \| + \alpha_n \| Tp - Tz_n \| \\
&\leq \| (1 - \alpha_n) \| x_n - p \| + \alpha_n[\varphi(\| p - Tp \|) + b \| p - z_n \|] \\
&= \| (1 - \alpha_n) \| x_n - p \| + b\alpha_n \| 1 - \beta_n)x_n + \beta_n Tx_n - p \| \\
&= \| (1 - \alpha_n) \| x_n - p \| + b\alpha_n[\| 1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p) \| \\
&\leq \| (1 - \alpha_n) \| x_n - p \| + b\alpha_n(1 - \beta_n) \| x_n - p \|) + b\alpha_n\beta_n) \| Tp - Tx_n \| \\
&\leq \| (1 - \alpha_n) \| x_n - p \| + b\alpha_n(1 - \beta_n) \| x_n - p \|) + b\alpha_n\beta_n[\varphi(\| p - Tp \|) + b \| p - x_n \|] \\
&= (1 - \alpha_n + b\alpha_n) \| x_n - p \| - b\alpha_n\beta_n(1 - b) \| x_n - p \| \\
&\leq [1 - (1 - b)\alpha_n] \| x_n - p \| \\
&= \exp(-(1 - b)\alpha_n) \| x_n - p \| \\
&\leq \exp(-(1 - b)\alpha_n)\exp(-(1 - b)\alpha_{n-1}) \| x_{n-1} - p \| \\
&\leq \exp(-(1 - b)\alpha_n)\exp(-(1 - b)\alpha_{n-1}) \ldots \leq \exp(-(1 - b)\alpha_o) \| x_o - p \| \\
&= \exp(-(1 - b)\textstyle\sum_{j=0}^{n} \alpha_j) \| x_o - p \| \to 0,
\end{aligned}$$

as $n \to \infty$.

Hence, inequality (5) yields $\lim_{n\to\infty} y_n = p$.

Conversely, suppose that $\lim_{n\to\infty} y_n = p$. Then,

$$\begin{aligned}
\varepsilon_n &= \| y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ts_n \| \\
&\leq \| y_{n+1} - p \| + \| p - (1 - \alpha_n)y_n - \alpha_n Ts_n \| \\
&= \| y_{n+1} - p \| + \| (1 - \alpha_n)(y_n - p) + \alpha n(Ts_n - p) \|
\end{aligned}$$

$\| y_{n+1}-p\| + (1-\alpha_n)\| y_n - p\| + \alpha_n \| Tp-Ts_n\|$
$\| y_{n+1}-p\| + (1-\alpha_n)\| y_n - p\| + \alpha_n(\varphi(\| p-Tp\|)$
$+ b\| p-s_n\|)$
$= \| y_{n+1}-p\| + (1-\alpha_n)\| y_n - p\| + b\alpha_n(1-\beta_n)(y_n - p)$
$+ \beta_n(Ty_n - p)\|$
$\| y_{n+1}-p\| + (1-\alpha_n)\| y_n - p\| + b\alpha_n(1-\beta_n)(y_n - p)$
$+ b\alpha_n\beta_n \| Tp - Ty_n\|$
$\| y_{n+1} - p\| + (1-\alpha_n + b\alpha_n - b\alpha_n\beta_n)\| (y_n - p) \|$
$+ b\alpha_n\beta_n (\varphi(\| p - Tp\| + b\| p - y_n\|))$
$= \| y_{n+1} - p\| + (1-\alpha_n + b\alpha_n - b\alpha_n\beta_n + b^2 n\beta_n)$
$\| (y_n - p) \| \to 0, \text{ as } n \to \infty$

This completes the proof of the Theorem.

Remark 3: Theorem 1 is a generalization of Theorem 2 of Osilike[1] and Theorem 30 of Rhoades[3]. If $\beta_n = 0, \forall n \geq 0$ in Theorem 1, we obtain a generalization of Theorem 2 of Rhoades[4] which itself is a generalization of both Theorem 3 of Harder and Hicks[2] and Theorem 2 of Rhoades[5].

By Remark 2, we have the following stability result for the Mann iteration process.

Corollary 1: Let $(E, \|.\|)$ be a normed linear space and let $T: E \to E$ be a selfmap of E satisfying the contractive definition (4). Suppose T has a fixed point p and let be the Mann iteration process satisfying the conditions of Remark 2. then, the Mann iteration process is T-stable.

Proof: The proof follows directly from Theorem 1, by putting $\beta_n = 0$.

Remark 4: Corollary 1 is a generalization of Theorem 2 of Rhoades[4], which itself is a generalization of both Theorem 3 of Harder and Hicks[2] and Theorem 2 of Rhoades[5].

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