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On a Problem Connected with Navier-stokes Equations in Non Cylindrical Domains

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Abstract: In this study we showed the existence of weak solutions of equations that represent flows of a non-homogeneous viscous incompressible fluids in a non cylindrical domain in R^3. The classical Navier-stokes equation is a particular case of the equations here considered.

Key words: Non Homogeneous Fluids, Navier-stokes, Non Cylindrical Domains

INTRODUCTION

Let T > 0 be a real number and $\{\ \ _t\}_{0\ \ t\ \ T}$ a family of bounded open subsets of R^n with boundary $\partial\ \ _t = \ \ _t$. Let us consider the non cylindrical domain $\hat{Q} = \bigcup_{0<t<T} (\Omega_t \times \{t\})$ whose lateral boundary $\hat{\Sigma} = \bigcup_{0<t<T} (\Gamma_t \times \{t\})$ is assumed to be regular. Consider the flows of viscous, incompressible and nonhomogeneous fluids in $\hat{Q}$. The non homogeneity of the fluids means that the density is a non constant function $\ \ = \ \ (x,t), (x,t) \in (\ \ _t \times \{t\})$. These flows are governed by the following system of Navier-stokes types equations.

$$\frac{\partial}{\partial t}(\rho u) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(u_j \rho u) - \mu \Delta u = \rho f - \nabla p \text{ in } \hat{Q} \quad (1)$$

$$\text{div } u = 0 \text{ in } \hat{Q} \quad (2)$$

$$\frac{\partial \rho}{\partial t} + (u.\nabla)\rho = 0 \text{ in } \hat{Q} \quad (3)$$

$$u = 0 \text{ on } \hat{\Sigma} \quad (4)$$

$$u(x,0) = u_0(x) \text{ in } \ \ _0 \quad (5)$$

$$\ \ (x,0) = \ \ _0(x) \text{ in } \ \ _0 \quad (6)$$

where, $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ is the velocity, $\ \ u = (\ \ u_1, \ldots, \ \ u_n)$, $\nabla u = (\nabla u_1, \ldots, \nabla u_n)$, $(u.\nabla)\ \ = u_j \frac{\partial \rho}{\partial x_j}$ (with the summation convention), μ is a positive constants, p(x,t) is the pressure and it is a real valued function, $\ \ (x, t)$ density of the fluid at point $(x,t) \in (\ \ _t \times \{t\})$ and $f = f(x,t)$ is the external force vector field. In this study we will consider weak solutions of the system(1-6) on certain non cylindrical domains under standard hypothesis on f and u_0 in the dimensions n=3, we also assume that

$$0 < \ \ \ \ _0(x) \ \ \ < \quad (7)$$

To define these domains les us consider K(t) a matrix valued function

$$[0,T] \quad R^{n^2}$$

$$t \mapsto K(t),$$

and $\ \ \subset R^n$ a bounded domain with smooth boundary and containing the origin. Let us define the family of sets

$$\ \ _t = \{x = K(t)y \ ; \ y \in \ \ \}.$$

and the respective non cylindrical domains $\hat{Q} = \bigcup_{0<t<T} (\Omega_t \times \{t\})$. Global existence results for such nonhomogeneous, incompressible Navier-stokes equation were first obtained by Kazhikov [1], Kazhikov and Smagulov [2], Antonzev and Kajikov [3], Antonzev, *et al.* [4] and Lions [5, 6] – in the case $0 < \ \ \ _0(x) \ \ \ < \ \ $, that is $\ \ _0$ has a lower bound positive and in context of cylindrical domains. These results were extended by various authors and in particular by Simon [7-9] allowing $\ \ _0$ to vanish. We also observe that Kim [10] has studied problem (1-6) for cylindrical domains under more regularity assumptions on the data (u_0, f), thus obtaining considerably more regular solutions. In the bi-dimensional case and still for cylindrical domains the existence and uniqueness of classical solutions, assuming sufficiently regular initial data, were obtained by Ladyshenkaya [11]. The first result for the system (1-6) in non cylindrical domains were obtained by Limaco [12]. Here we are considering the same equations as described by Limaco [12], however, in more general non cylindrical domains. By a suitable change of variable, we transform the non cylindrical problem (1-6) into a problem defined in the cylinder Q = × (0,T). In Q we follow the ideas of Lions [5, 6].

NOTATION AND MAIN RESULTS

To show our main result we assume the following hypothesis in K(t).

(H1) K(t) = k(t) M, where k: [0,T] IR, $k \in C^1([0,T])$, k(t) > 0 and M is an invertible n by n matrix whose entries are real constants.

Consider the notation

$K(t) = (\ _{ij}(t))$ and $K \ ^1(t) = (\ _{ij}(t))$.

By C we represent several positive constants. In order to transform the non cylindrical problem (1-6) into a new problem in the cylindrical domain Q, we introduce the functions

$u(x,t) = v(K\ ^1(t)x,t)$, $f(x,t) = \ g(K\ ^1(t)x,t)$
$p(x,t) = q(K\ ^1(t)x,t)$, $(x,t) = \ (K\ ^1(t)x,t)$
$u_0(x) = v_0(K\ ^1(0)x)$, $_0(x) = \ _0(K\ ^1(0)x)$.

We have the following identity

$x_r = \ _{rj}y_j$, $y_l = \ _{lr}x_r$ and $\frac{\partial y_l}{\partial t} = \beta'_{lr}x_r$, or

$$\frac{\partial y_l}{\partial t} = \beta'_{lr}\alpha_{rj}y_j$$

Since $\frac{y_l}{x_j} = \ _{lj}$, we obtain

$$\frac{\partial u_i}{\partial x_k}(x,t) = \beta_{jk}\frac{\partial v_i}{\partial y_j}(y,t) \tag{8}$$

Also

$$\frac{\partial^2 u_i}{\partial x_k^2}(x,t) = \beta_{jk}\beta_{rk}\frac{\partial^2 v_i}{\partial y_r \partial y_j}(y,t)$$

Consequently

$$\Delta u_i(x,t) = \beta_{jk}\beta_{rk}\frac{\partial^2 v_i}{\partial y_j \partial y_r}(y,t)\cdot$$

We have also

$$\frac{\partial p}{\partial x_i} = \frac{\partial q}{\partial y_j}\beta_{ji}$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y_j}\beta'_{jl}x_l$$

and

$$\frac{\partial(\rho u)}{\partial t} = \frac{\partial(\varphi v)}{\partial t}(y,t) + \frac{\partial(\varphi v)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k\cdot$$

Remark 1: If we have $_0(x) = \ _0$ = constant say $_0 = 1$, then = 1 satisfies (3,4 and 6) and the problem reduces to the classical Navier-stokes situations in the cylindrical domain.

Remark 2: since div u = 0, (3) in equivalent to $\frac{\partial \rho}{\partial t}$ + div(u) = 0.

Then, from (1-6) it follows that

$$\frac{\partial(\varphi u)}{\partial t} - \mu\frac{\partial}{\partial y_j}\left(a_{jr}(t)\frac{\partial v}{\partial y_j}\right) + \frac{\partial}{\partial y_j}(v_i \varphi v)\beta_{ji} + \frac{\partial(\varphi v)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k$$

$$= \varphi g - \left(\frac{\partial q}{\partial y_j}\beta_{j1}, \ldots, \frac{\partial q}{\partial y_j}\beta_{jn}\right)^t \text{ in Q} \tag{9}$$

$$\text{div}(M\ ^1 v^t) = 0 \text{ in Q} \tag{10}$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial(\varphi v_i)}{\partial y_j}\beta_{ji} + \frac{\partial \varphi}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k = 0 \text{ in Q} \tag{11}$$

$$v = 0 \text{ on } \Sigma \tag{12}$$

$$v(y,0) = v_0(y) \text{ on} \tag{13}$$

$$(y,0) = \ _0(y) \text{ on } \ . \tag{14}$$

Here $a_{jr} = \ _{jk} \ _{rk}$ and v^t is the transposed of the row vector $v = (v_1,\ldots,v_n)$ and Q = × (0,T), = × (0,T).

Remarks 3: It follows of (8) that

$$\frac{\partial u_i}{\partial x_i}(x,t) = \sum_j \beta_{ji}\frac{\partial v_i}{\partial y_j}(y,t), \text{ i fixed.}$$

Therefore, div u(x,t) = $_{ji}\frac{\partial v_i}{\partial y_j}(y,t)$.

On the other hand K(t) = k(t)M. Then $K\ ^1(t) = \frac{1}{k(t)}M\ ^1$.

Therefore

$_{ij} = \frac{1}{k}\eta_{ij}$ where, $M\ ^1 = (\ _{ij})$. Thus,

$$\text{div } u(x,t) = \frac{1}{k(t)}\text{div}(M\ ^1 v^t(y,t))$$

and by (2), (10) follows.

Remark 4: Let A(t) be the operator

$$A(t)v = -\frac{\partial}{\partial y_r}\left(a_{jr}(t)\frac{\partial v}{\partial y_j}\right),\ \ v\in (H_0^1(\Omega))^n \qquad (15)$$

We showed in the Lemma 1 that A(t) is uniformly elliptic in [0,T].

To state the main result we introduce some space. Let $_t$ be the space

$_t$ = { ∈(D($_t$))^n; div = 0}

and V_s($_t$) be the closure of $_t$ in the space $(H^s($ $_t))^n$ where, s is a nonnegative real number. We use the particular notation

V_1($_t$) = V($_t$) and V_0($_t$) = H($_t$)

The inner product of this spaces are denoted, respectively by $(u,z)_{H(\ t)}$ and $((u,z))_{V(\ t)}$. Then for u = $(u_1,\ldots u_n)$ and z = $z(z_1\ldots..z_n)$ we have

$$(u,z)_{H(\Omega_t)} = \int_{\Omega_t} u_i(x)z_i(x)dx\,,$$

$$((u,z))_{V(\Omega_t)} = \int_{\Omega_t} \frac{\partial u_i}{\partial x_j}(x)\frac{\partial z_i}{\partial x_j}(x)dx.$$

Note that Vs($_t$) is continuously embedded in $(H_0^1(\Omega_t))^n$ for s ≥ n/2 and

$V($ $_t) = \{u \in (H_0^1(\Omega_t))^n\ ;\ \text{div } u = 0\}$.

For these results by Lions [13]. In similar way we introduce the space Vs( ). In this case has the form

= {φ ∈ (D( ))^n; div(M ^{1}φ^t) = 0}.

We consider the particular notations V_1( ) = V, V_0( ) = H and $(v,w)_H$ = (v,w), $((u,v))_V$ = ((v,w)), $|v|_H$ = |v| and $||v||_V$ = ||v||.

The spaces $L^p(0,\ T,\ V_s($ $_t))$ are defined in Appendix. Following the ideas of Lions [5,6], we define the notion of weak solutions (for n=3) of problem (1-6) and (9-14).

Non cylindrical Case: Find u(x,t) and (x,t) such that

u ∈ $L^2(0, T, V($ $_t))$ $L^2(0, T, H($ $_t))$, ∈ $L^\infty(\hat{Q})$

$$-\int_{\hat{Q}}\rho u\frac{\partial\phi}{\partial t}dx\,dt - \mu\int_{\hat{Q}}\nabla u\nabla\phi dx\,dt - \sum_{i,j=1}^{3}\int_{\hat{Q}} u_j\rho u_i\frac{\partial\phi_i}{\partial x_j}dx\,dt = (16)$$
$$-\int_{\hat{Q}}\rho f\phi dxdt + \int_{\Omega_0}\rho_0 u_0\phi(0)dx$$

$$-\int_{\hat{Q}}\rho\frac{\partial\xi}{\partial t}dx\,dt - \sum_{i=1}^{3}\int_{\hat{Q}}\rho u_i\frac{\partial\xi}{\partial x_i}dx\,dt = \int_{\Omega_0}\rho_0\xi(0)dx \qquad (17)$$

for all φ, ξ∈ $[C^1(\bar{Q})]^3$ with compact support contained in $\hat{Q}$ ∪ { $_0$ × {0}} and divφ = 0.

Cylindrical Case: Find and such that
∈ $L^2(0, T, V)$ $L^\infty(0, T, H)$, ∈ $L^\infty($ × (0,T))

$$-\int_Q \varphi v\frac{\partial\psi}{\partial t}dy\,dt + \mu\int_Q a_{jl}(t)\frac{\partial v}{\partial y_j}\frac{\partial\psi}{\partial y_l}dy\,dt + \int_Q \beta_{ji}v_i\varphi v\frac{\partial\psi}{\partial y_j}dydt + \qquad (18)$$
$$\int_Q \beta'_{ji}\alpha_{lk}\varphi v\frac{\partial(\psi yk)}{\partial y_j}dydt = \int_Q \varphi g\psi dydt - \int_\Omega \varphi_0 v_0\psi(0)dy$$

for all ∈ $[C^1(\bar{\Omega}\times[0,T])]^3$ with compact support contained in × (0,T) and div(M ^{1} ^{t})=0.

$$\frac{\partial\varphi}{\partial t} + \frac{\partial(\varphi v_i)}{\partial y_j}\beta_{ji} + \frac{\partial\varphi}{\partial y_j}\ \ '_{jl}\ \ _{lk}y_k = 0 \qquad \text{in Q} \quad (19)$$

(x,0) = $_0$(x) in (20)

Next we shall state the main results of this study. Let $\hat{Q}$ and $\hat{\Sigma}$ be as in the section and n = 3. We have

Theorem 1: Assume that hypothesis (H1) is satisfied and that ^{0} satisfies (7).

If f ∈ $L^2(0, T, H($ $_t))$, ^{0} ∈ $L^\infty($ $_0)$ and u_0∈ H($_0$), then there exists a weak solution of the problem (1-6).

The theorem 1 is consequence of the following two results:

Theorem 2: If g∈ $L^2(0,T,H)$, v_0∈ H and $_0$ ∈ L^∞( ) then there exists a weak solution of the problem (9-14).

Theorem 3: Problems (1-6) and (9-14) are equivalent.

Remark 5: Uniqueness is an open question. It is still open in the particular case $_0$(x) = $_0$ with a constant and in context of cylindrical or noncylindrical domain.

PROOF OF RESULTS

Proof of Theorem 2: We use Semi-Galerkin method. We consider as approximation of (9-14) which is of the Galerkin's type in v and where in (11) we replace v by its approximation (hence the terminology of Semi-Galerkin). For this, we may consider a family of internal approximation V_m ⊂ V such that V_m is a subspace of V dimension m,

∀v ∈ V, there exist a sequence v_m ∈ V such that v_m v in V as m ∞.

We also may assume that all components of functions v ∈ V_m belong to $C^1(\overline{\Omega})$. Because this, we can consider a basis (w^i) of V such that $w^i \in (C^1(\overline{\Omega}))^3$. Note that the embedding $C^1(\overline{\Omega})$ V ⊂ V is dense, continuous and $C^1(\overline{\Omega})$ V is separable. Then there exists (w^i).

Let $V_m = [w^1, \ldots, w^m]$, $v_m = \sum_{i=1}^{m} g_{im}(t) w^i$ with $v_m \in C^1([0,T_m], V_m)$ and

$_m \in C^1([0,T_m], C^1(\overline{\Omega}))$ satisfying

$$\left(\frac{\partial}{\partial t}(\varphi_m v_m), w^k\right) + \mu\left(a_{jl}(t)\frac{\partial v_m}{\partial y_j}, \frac{\partial w^k}{\partial y_l}\right) + \left(\frac{\partial}{\partial y_j}(v_{m_i}\varphi_m v_m)\beta_{ji}, w^k\right) + \tag{21}$$
$$\left(\frac{\partial(\varphi_m v_m)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, w^k\right) = (\varphi_m g, w^k), \qquad k = 1,\ldots,m$$

$$\frac{\partial \varphi_m}{\partial t} + \frac{\partial(\varphi_m v_{m_i})}{\partial y_j}\beta_{jl} + \frac{\partial \varphi_m}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k = 0 \tag{22}$$

$$v_m(0) = v_{0m} \quad v_0 \text{ in } V \tag{23}$$

$$(y,0) = {}_{om}(y) \quad {}_0(y) \text{ in } L^q(\ ), \tag{24}$$

where, $1 < q < +\infty$, $< {}_{om} <$

Local Existence of v_m and $_m$: Assuming u_m to be known, we can express the solution $_m(t,y)$ of (22), (24) as follows:

$_m(t,y) = {}_{0m}(x_m(0,t,y))$

where,
$x_m(\delta,t,y) = (x_m^1(\delta,t,y), x_m^2(\delta,t,y), x_m^3(\delta,t,y))$ is the solution of

$$\begin{cases} \dfrac{dx_m^1}{ds}(\delta,t,y) = \beta_{1i} v_{m_i}(x_m(\delta,t,y),\delta) + \beta'_{1l}\alpha_{lk}x_m^k(\delta,t,y) \\ \dfrac{dx_m^2}{ds}(\delta,t,y) = \beta_{2i} v_{m_i}(x_m(\delta,t,y),\delta) + \beta'_{2l}\alpha_{lk}x_m^k(\delta,t,y) \\ \dfrac{dx_m^3}{ds}(\delta,t,y) = \beta_{3i} v_{m_i}(x_m(\delta,t,y),\delta) + \beta'_{3l}\alpha_{lk}x_m^k(\delta,t,y) \end{cases}$$

$x_m(t,t,y) = y$.

We multiply (22) by $v_m w^k$ and we integrate over and we add the result to (21), we obtain

$$\int_\Omega \left(\varphi_m \frac{\partial v_m}{\partial t} + v_{m_i}\varphi_m \frac{\partial v_m}{\partial y_j}\beta_{ji} + \varphi_m \frac{\partial v_m}{\partial y_j}\beta'_{jl}\alpha_{ls}y_s\right) w^k dx + \tag{25}$$
$$\mu\left(a_{jl}\frac{\partial v_m}{\partial y_j}, \frac{\partial w^k}{\partial y_l}\right) = 0$$

Since $v_m(t) = \sum_{r=1}^{m} g_m(t) w^k$ then (25) has a form

$$Q_{rk}(\varphi_m)\frac{dg_{rm}}{dt} + \Pr(g_{1m},\ldots,g_{mm}) = 0 \tag{26}$$

where, $Q_{rk}(\ _m) = \int_\Omega \varphi_m(y,t) w^r w^k dy$, $_m(y,t) = {}_{0m}(x_m(0,t,y))$ and $0 < \ < {}_m(y,t) < \ $.

We have that $\sqrt{\varphi_m} w^i$ are linearly independent. Indeed, if $\sum_{i=1}^{m} \lambda_i \sqrt{\varphi_m} w^i = 0$ then $\sum_{i=1}^{m} \lambda_i w^i = 0$ and hence $\lambda_i = 0$, $i = 1,\ldots, m$. Thus $Q_{rk}(\ _m) = \left(\sqrt{\varphi_m} w^r, \sqrt{\varphi_m} w^k\right)$ is non singular. Then (26) is equivalent to

$$\frac{dg_{rm}}{dt} = -Q_{rk}^{-1}(\varphi_m)\Pr(g_{1m},\ldots,g_{mm}). \tag{27}$$

Where, P_r, Q_{rk}^{-1} are differentiable continuous. Then the system of nonlinear differential equation (27) has a local solution, i.e., v_m and $_m$ are solutions of (21-24). The standard a priori estimates, which follow prove the global existence of the solution v_m, $_m$. The extention of the solution to the whole interval [0,T] is consequence of the estimates which follows.

Estimate I: If we take $w^k = 2v_m$ in (21) and if we multiply (22) by $-|v_m|_{R^n}^2$ (where $|.|_{R^n}^2$ denote the usual the norm in R^n, n = 3), we obtain after adding up

$$2\left(\frac{\partial(\varphi_m v_m)}{\partial t}, v_m\right) - \left(\frac{\partial \varphi_m}{\partial t}, |v_m|_{R^n}^2\right) + \mu\left|\beta ji \frac{\partial v_{m_i}}{\partial y_j}\right|_{L^2(\Omega)}^2 +$$
$$2\left(\frac{\partial}{\partial y_j}(v_{m_i}\varphi_m v_m)\beta_{ji}, v_m\right) - \left(\frac{\partial(\varphi_m v_{m_i})}{\partial y_j}\beta_{ji}, |v_m|_{R^n}^2\right) - \tag{28}$$
$$\left(\frac{\partial(\varphi_m)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, |v_m|_{R^n}^2\right) + 2\left(\frac{\partial(\varphi_m v_m)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, v_m\right) = 2(\varphi_m g, v_m).$$

Direct calculation shows that:

$$2\left(\frac{\partial(\varphi_m v_m)}{\partial t}, v_m\right) - \left(\frac{\partial \varphi_m}{\partial t}, |v_m|_{R^n}^2\right) = \frac{d}{dt}\int_\Omega \varphi_m |v_m|_{R^n}^2 dy \tag{29}$$

$$2\left(\frac{\partial}{\partial y_j}(v_{m_i}\varphi_m v_m)\beta_{ji}, v_m\right) - \left(\frac{\partial(\varphi_m v_{m_i})}{\partial y_j}\beta_{ji}, |v_m|_{R^n}^2\right) =$$
$$\left(\frac{\partial(\varphi_m v_{m_i})}{\partial y_j}\beta_{ji}, |v_m|_{R^n}^2\right) + 2\left(v_{m_i}\varphi_m \frac{\partial v_m}{\partial y_j}\beta_{ji} v_m\right) = \tag{30}$$
$$\left(\frac{\partial(\varphi_m v_{m_i})}{\partial y_j}\beta_{ji}, |v_m|_{R^n}^2\right) + \left(v_{m_i}\varphi_m\beta_{ji}\frac{\partial}{\partial y_j}|v_m|_{R^n}^2\right) =$$
$$\int_\Omega \frac{\partial}{\partial y_j}(v_{m_i}\varphi_m |v_m|^2)\beta_{ji} dy = \int_\Omega (v_{m_i}\beta_{ji}\varphi_m |v_m|^2)\eta_i dy = 0$$

by Gauss Theorem and by virtue of div $v_m = 0$ (we denote of = (1,…, n) the unit outward normal vector to). Also,

$$I_1 = -\left(\frac{\partial \varphi_m}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, |v_m|^2_{R^n}\right) + 2\left(\frac{\partial(\varphi_m v_m)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, v_m\right) =$$
$$-\left(\frac{\partial \varphi_m}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, |v_m|^2_{R^n}\right) + 2\left(\frac{\partial \varphi_m}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, |v_m|^2_{R^n}\right) + \quad (31)$$
$$\left(\varphi_m\beta'_{jl}\alpha_{lk}y_k, \frac{\partial}{\partial y_j}|v_m|^2_{R^n}\right) = \left(\frac{\partial}{\partial y_j}(\varphi_m |v_m|^2_{R^n}), \beta'_{jl}\alpha_{lk}y_k\right) =$$
$$-\int_\Omega \varphi_m |v_m|^2 \beta'_{jl}\alpha_{lk}\delta_{kj}dy = -\int_\Omega \varphi_m |v_m|^2)\beta'_{kl}\alpha_{lk}dy$$

where, kj is the Kronecher symbol.

We observe that, in view of m(y,t) = 0,m = (x_m(0,t,y)), (H1) and (7), we obtain

$$|I_1| \le C\int_\Omega \varphi_m |v_m|^2\, dy \quad (32)$$

$$0 < \quad < \;_m(y,t) < \quad (33)$$

In view of (29-33), we obtain of (28) that

$$\frac{d}{dt}\int_\Omega \varphi_m |v_m|^2_{\mathbb{R}^n}\, dy + C_0 \|v_m\|^2 \le \int_\Omega |g|^2 dy + C\int_\Omega \varphi_m |v_m|^2_{\mathbb{R}^n}\, dy$$

By (33) and Gronwall inequality, we have

$$\alpha |v_m|^2 + C\int_0^T \|v_m(s)\|^2\, ds \le C \quad (34)$$

Using (33) and (34) we obtain that v_m and m are defined to [0,T] and

(v_m) is bounded in $L^2(0,T,V)$ L (0,T,H) (35)

Estimate II: Considering $v = w^k \in V_m$ in (21) we have

$$\left(\frac{\partial}{\partial t}(\varphi_m v_m), v\right) = (J_m(t), v) \quad (36)$$

where:

$$(J_m(t), v) = -\mu\left(a_{jl}(t)\frac{\partial v_m}{\partial y_j}, \frac{\partial v}{\partial y_l}\right) - \left(\frac{\partial}{\partial y_j}(v_{m_i}\varphi_m v_m)\beta_{ji}, v\right) - \quad (37)$$
$$\left(\frac{\partial(\varphi_m v_m)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, v\right) + (\varphi_m g, v)$$

Integrating (37) over (t,t +) with t + T and v fixed we obtain

$$(\varphi_m(t+\delta)v_m(t+\delta) - \varphi_m(t)v_m(t), v) = \left(\int_t^{t+\delta} J_m(s)ds, v\right) \quad (38)$$

We take now $v = v_m(t + \;) - v_m(t)$ in (38). Let us set:

X_m=(m(t+)(v_m(t+)–v_m(t)), v_m(t+)–v_m(t)) (39)

Y_m=((m(t+)– m(t))v_m(t), v_m(t+)–v_m(t)) (40)

Follows of (38) that

$$X_m + Y_m = \left(\int_t^{t+\delta} J_m(s)ds, v_m(t+\delta) - v_m(t)\right) \quad (41)$$

Since m we have

X_m $|v_m(t + \;) - v_m(t)|^2$ (42)

Let us now transform Y_m. It follows from (22) that

$$((\;_m(t+\;) - \;_m(t))v_m(t), v) =$$
$$-\int_\Omega\left[\int_t^{t+\delta}\frac{\partial(\varphi_m v_{m_i})}{\partial y_j}\beta_{ji}ds\right]v_m v\, dy - \int_\Omega\left[\int_t^{t+\delta}\frac{\partial \varphi_m}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k ds\right]v_m v\, dy$$
$$= -\int_\Omega \frac{\partial}{\partial y_j}\left[\int_t^{t+\delta}\varphi_m v_{m_i}\beta_{ji}ds\right]v_m v\, dy - \int_\Omega \frac{\partial}{\partial y_j}\left[\int_t^{t+\delta}\varphi_m\beta'_{jl}\alpha_{lk}ds\right]y_k v_m v\, dy \quad (43)$$
$$= \int_\Omega\left[\int_t^{t+\delta}\varphi_m v_{m_i}\beta_{ji}ds\right]\left(\frac{\partial v_m}{\partial y_j}v + v_m\frac{\partial v}{\partial y_j}\right)dy +$$
$$+\int_\Omega\left[\int_t^{t+\delta}\varphi_m\beta'_{jl}\alpha_{lk}ds\right]\left(\frac{\partial y_k}{\partial y_j}v_m v + y_k\frac{\partial v_m}{\partial y_j}v + y_k v_m\frac{\partial v}{\partial y_j}\right)dy$$

Since, in particular, the embedding $V \subset (L^4(\;))^3$ is continuous (n = 3), 0 < < m < and ij, ij ∈ $C^1([0,T])$ it follows that

$$\left|\int_\Omega(\varphi_m(t+\delta) - \varphi_m(t))v_m(t)v\, dy\right| \le$$
$$\le C\left(\int_t^{t+\delta}|v_m(s)|_{L^4(\Omega)}ds\right)(\|v_m(t)\|\,\|v\|_{L^4(\Omega)} + |v_m(t)|_{L^4(\Omega)}\|v\|)$$
$$+C\delta\|v_m(t)\|\,\|v\| \quad (44)$$
$$\le C\left(\int_t^{t+\delta}\|v_m(s)\|ds\right)\|v_m(t)\|\,\|v\| + C\delta\|v_m(t)\|\,\|v\|$$
$$\le C\sqrt{\delta}\left(\int_t^{t+\delta}\|v_m(s)\|^2 ds\right)^{1/2}\|v_m(t)\|\,\|v\| + C\delta\|v_m(t)\|\,\|v\|.$$

Taking $v = v_m(t + \;) - v_m(t)$ in (44) and using (35), we have

$$|Y_m| \quad C\sqrt{\delta}\; (\|v_m(t)\|^2 + \|v_m(t+\;)\|^2. \quad (45)$$

We integrate (45) on (0,T –) and we use again (35), we obtain

$$\int_0^{T-\delta}|Y_m|\, dt \le C\sqrt{\delta} \quad (46)$$

We now estimate the term of the right side of (41). We have of (18) that

$$|(J_m(t), v)| \le \mu\left|\left(a_{jl}(t)\frac{\partial v_m}{\partial y_j}, \frac{\partial v}{\partial y_l}\right)\right| + \left|\left(\frac{\partial}{\partial y_j}(v_{m_i}\varphi_m v_m), v\right)\right| + \quad (47)$$
$$\left|\left(\frac{\partial(\varphi_m v_m)}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k, v\right)\right| + |(\varphi_m g, v)|$$

In analogy of (43,44), we obtain

$$|(J_m(t),v)| \le C|g(t)|\,\|v\| + C\|v_m\|\,\|v\| + C\|v_m\|^2_{(L^4(\Omega))^3}\|v\| \le C(|g(t)| + \|v_m\| + \|v_m\|^2)\|v\| \quad (48)$$

Then

$$\left|\left(\int_t^{t+\delta} J_m(s)ds, v_m(t+\delta)-v_m(t)\right)\right| \le C\left(\int_t^{t+\delta}|g(s)|+\|v_m(s)\|+\|v_m(s)\|^2\right)(\|v_m(t)\|+\|v_m(t+\delta)\|) \quad (49)$$

We integrate (49) on (0, T –); we obtain

$$\int_0^{T-\delta}\left|\left(\int_t^{t+\delta}J_m(s)ds, v_m(t+\delta)-v_m(t)\right)\right|dt \le C\int_0^{T-\delta}\int_t^{t+\delta}(|g(s)|+\|v_m(s)\|+\|v_m(s)\|^2)(\|v_m(t)\|+\|v_m(t+\delta)\|)ds\,dt \quad (50)$$

By Fubini Theorem and with the convention that $v_m = 0$ on (– ,0), we have

$$\begin{aligned}
&\int_0^{T-\delta}\left|\left(\int_t^{t+\delta}J_m(s)ds, v_m(t+\delta)-v_m(t)\right)\right|dt \le \\
&C\int_0^T\left(\int_{s-\delta}^s|g(s)|+\|v_m(s)\|+\|v_m(s)\|^2\right)(\|v_m(t)\|+\|v_m(t+\delta)\|)dt\,ds \le \\
&C\int_0^T(|g(s)|+\|v_m(s)\|+\|v_m(s)\|^2)ds\int_{s-\delta}^s(\|v_m(t)\|+\|v_m(t+\delta)\|)dt \le \\
&C\left(\int_{s-\delta}^s\|v_m(t)\|dt+\int_{s-\delta}^s\|v_m(t+\delta)\|dt\right) \le \\
&C\sqrt{\delta}\left[\left(\int_{s-\delta}^s\|v_m(t)\|^2dt\right)^{1/2}+\left(\int_{s-\delta}^s\|v_m(t+\delta)\|^2dt\right)^{1/2}\right] \le C\sqrt{\delta}
\end{aligned} \quad (51)$$

as above.
In view of (41,42,46 and 51) it follow that

$$\alpha\int_0^{T-\delta}|v_m(t+\delta)-v_m(t)|^2dt \le \int_0^{T-\delta}X_m dt \le \int_0^{T-\delta}|Y_m|dt + \int_0^{T-\delta}\left|\left(\int_t^{t+\delta}J_m(s)ds, v_m(t+\delta)-v_m(t)\right)\right|dt \le c\sqrt{\delta} \quad (52)$$

for all > 0, 0 < < T.
In virtue of the a priori estimate and by a compactness result (Appendix), Lemma 2, we can extract subsequences, still demoted by v_m and $_m$ such that

$v_m \xrightarrow{*}$ u weakly in $L^2(0,T,V)$ (53)

$v_m \longrightarrow$ u weakly-star in L (0,T,H) (54)

v_m u strongly in $L^p(0,T,(L^q( ))^3)$ (55)

$_m \xrightarrow{*}$ weak-star in L (Q) (56)

where, p∈(2, +), q ∈ (2,6) and $\frac{1}{q}+\frac{3}{2q}>\frac{3}{4}$

We also know that

(y,t) < (57)

It follows from (55) and (56) that

$_{ji}v_{mi}\ _m \longrightarrow\ _{ji}v_i$ weakly in $L^2(0,T,L^2( ))$ (58)

and then

$$\beta_{ji}\frac{\partial}{\partial y_j}(u_{m_i},\varphi_m) \ * \ \beta_{ji}\frac{\partial}{\partial y_j}(v_i\varphi)$$

weak-star in $L^2(0, T, H^{1}( ))$ (59)

By analogy

$$\frac{\partial\varphi_m}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k \xrightarrow{*} \frac{\partial\varphi}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k$$

weak-star in $L^2(0, T, H^{1}( ))$ (60)

Thus of (59), (60) and (22) we obtain

$$\frac{\partial\varphi_m}{\partial t} \xrightarrow{*} \frac{\partial\varphi}{\partial t} \text{ weak-star in } L^2(0,T,H^{1}( )) \quad (61)$$

The equation (22) gives in the limit

$$\frac{\partial\varphi}{\partial t}+\frac{\partial(\varphi v_i)}{\partial y_j}\beta_{ji}+\frac{\partial\varphi}{\partial y_j}\beta'_{jl}\alpha_{lk}y_k = 0$$

in the sense of $L^2(0,T,H^{-1}( ))$.

It follows from (56) and (61) that in particular $_m(y,0)$ (y,0) in $H^{1}( )$

and therefore, (y,0) = $_0(y)$

Taking p = q = 3 in (55) we deduce that

$$v_{m_i}v_{m_l}\varphi_m\beta_{ji} \longrightarrow v_i v_l\varphi\beta_{ji}$$

weakly in $L^{3/2}(0,T,L^{3/2}( ))$.
Also we deduce of (55), with p = q = 2 and (56) that

$$\varphi_m v_{m_l}\beta'_{jl}\alpha_{lk} \longrightarrow \varphi v_l\beta'_{jl}\alpha_{lk} \text{ weakly in } L^2(0,T,L^2( )).$$

These convergence and density argument permit to pass to the limit in (21) and (18) is verified. This conclude the proof of Theorem 2.

Proof of Theorem 3: Recall that x = K(t)y, y = K^{-1}(t)x, $x_r = \ _{rj}y_j$, $y_l = \ _{lr}x_r$. We have established that u(x,t) = v(K^{-1}(t)x,t) and (x,t) = (K ^{1}(t)x,t). Let us consider ϕ ∈ $(C^1(\bar{\hat{\Omega}}))^3$ with compact support in $\hat{Q}$ ∪{ × {0}} and divϕ = 0. We define.
(y,t) = |det K(t)| ϕ(K(t)y,t) or equivalently

$\phi(x,t) = |\det K\ ^1(t)|\quad (K\ ^1(t)x,t)$

where, det K denotes the determinant of the matrix K.

It is easy see that $\quad \in (C^1(\overline{\Omega}\times(0,T)))^3$ with compact support in $\quad\times[0,T)$ and div(M $^1\quad ^t$) = 0 (Remark 3).

We have the following identity

$$\frac{\partial\phi_i}{\partial t}(x,t) = |\det K^{-1}(t)|\frac{\partial\psi_i}{\partial t}(y,t) + |\det K^{-1}(t)|'\psi_i(y,t) + |\det K^{-1}(t)|\frac{\partial\psi_i}{\partial y_j}\beta_{jl}\alpha_{lk}y_k \quad (62)$$

We have det K(t) = k(t)^n det M and hence

$$|\det K^{-1}(t)|' = -n\frac{k'(t)}{k(t)}|\det K^{-1}(t)| \quad (63)$$

On the other hand,

$$\beta'_{jl}\alpha_{lj} = \mathrm{tr}\left(\left(K^{-1}(t)\right)' K(t)\right) = \mathrm{tr}\left(-\frac{k'(t)}{k(t)}I\right) = -n\frac{k'(t)}{k(t)} \quad (64)$$

where, tr K denote the trace of the matrix K and I the identity matrix.

Combining (62), (63) and (64), we get

$$\frac{\partial\phi_i}{\partial t}(x,t) = |\det K^{-1}(t)|\left(\frac{\partial\psi_i}{\partial t} + \beta'_{jl}\alpha_{lk}\frac{\partial(\psi_i y_k)}{\partial y_j}\right)$$

Then

$$-\int_0^T\int_\Omega \varphi v\left(\frac{\partial\psi}{\partial t} + \beta'_{jl}\alpha_{lk}\frac{\partial(\psi y_k)}{\partial y_j}\right)dy\,dt = -\int_{\hat{Q}}\rho u\frac{\partial\phi}{\partial t}dx\,dt \quad (65)$$

Also we get

$$\int_0^T\int_\Omega a_{jl}(t)\frac{\partial v}{\partial y_j}\frac{\partial\psi}{\partial y_l}dy\,dt = \int_{\hat{Q}}\nabla u\nabla\phi\,dx\,dt \quad (66)$$

$$\int_0^T\int_\Omega \beta_{ji}v_i\varphi v\frac{\partial\psi}{\partial y_j}dy\,dt = \int_{\hat{Q}}u_j\rho u_i\frac{\partial\phi_i}{\partial x_j}dx\,dt \quad (67)$$

$$\int_0^T\int_\Omega \varphi g\psi\,dy\,dt = \int_{\hat{Q}}\rho f\,\phi\,dx\,dt \quad (68)$$

$$\int_\Omega \varphi_0 v_0\psi(0)\,dy\,dt = \int_{\Omega_0}\rho_0 u_0\phi(0)\,dx\,dt \quad (69)$$

and in view of (65-69) and (18) we obtain (16).

Now we assume (19) and prove (17). Indeed, let $\quad\in C^1(\overline{Q})$ with compact support in $\hat{Q}\cup\{\ \ _0\times\{0\}\}$. Let h(y,t) = |det K(t)| $\quad$(K(t)y,t). Then $h\in C^1(\overline{\Omega}\times[0,T])$ and h has compact support in $\quad\times[0,T]$. We note that the equality (19) is in sense of $L^2(0,T,H\ ^1(\quad))$. We multiply (19) by h and we integrate in $\quad\times[0,T]$; we obtain

$$-\int_0^T\int_\Omega \varphi\frac{\partial h}{\partial t}dy\,dt + \int_0^T\int_\Omega \varphi v_i\beta_{ji}\frac{\partial h}{\partial y_j}dy\,dt + \int_0^T\int_\Omega \varphi\beta'_{jl}\alpha_{lk}\frac{\partial(hy_k)}{\partial y_j}dy\,dt = 0$$

Similarly as in Theorem 2 we obtain (17). Moreover $u\in L^2(0,T,V(\ _t))\quad L\ (0,T,H(\ _t))$ and $\quad\in L\ (\hat{Q})$

Appendix: For the sake of completeness we will show some auxiliary results and also we will define some spaces used in our work. In order, let u(x,t) and $\quad$(y,t) be vector real functions related by:

$$u(x,t) = |\det K\ ^1(t)|\quad (K\ ^1(t)x,t). \quad (70)$$

We have

$$|u(t)|^2_{V(\Omega_t)} = \sum_{i,j,l=1}^n \int_\Omega |\det K^{-1}(t)|\left(\beta_{lj}(t)\frac{\partial\xi_i}{\partial y_l}\right)^2 dy. \quad (71)$$

This implies

$$c_1\|\xi(t)\|v \le \|u(t)\|v_{(\Omega_t)} \le c_2\|\xi(t)\|v, \quad (72)$$

where, c_1 and c_2 are constants independents of u and $\quad$. Motivate by (71) and (72) we define $L^p(0,T,V(\ \ t))$, (1 $\quad$ p $\quad$) as the space of (classes of) functions u: $\hat{Q}$ $\quad$ R such that there exists $\quad\in L^p(0,T,V(\ \ t))$ verifying (70) equipped with the norm

$$\|u\|_{L^p(0,T,V(\Omega_t))} = \left(\int_0^T\|u\|^p_{L^p(0,T,V(\Omega_t))}dt\right)^{1/p},\ 1\quad p< \quad (73)$$

$$\|u\|_{L^\infty(0,T,V(\Omega_t))} = \text{ess sup}_{t\in[0,T]}\|u\|_{V(\Omega_t)} \quad (74)$$

In a similar way we define the space $L^p(0,T,(\ \ _t))$. Let u(x,t) and w(y,t) be vector real functions such that

$$u(x,t) = w(K\ ^1(t)x,t). \quad (75)$$

Then

$$\frac{\partial u(x,t)}{\partial t} = \beta'_{lr}\alpha_{rj}y_j\frac{\partial w(y,t)}{\partial y_l} + \frac{\partial w(y,t)}{\partial t}. \quad (76)$$

Take $u\in L^2(0,T,V(\ _t))$. Then w verifying (75) belongs to $L^2(0,T,V)$. Motivated by (76) we say that $u\in L^2(0,T,H(\ _t))$ if $w\in L^2(0,T,H)$. In a similar way we say that $u\in L^2(0,T,V(\ _t))$ if $w\in L^2(0,T,V)$. In this case $u\in L^2(0,T,V_2(\ _t))$.

Concerning the operator defined by (15), we have

Lemma 1: Let A(t) the operator defined by (15) and a(t,v,w) the bilinear form defined by:

$$a(t, v, w) = \int_{\Omega} a_{rj}(t) \frac{\partial v_i}{\partial y_j} \frac{\partial w_i}{\partial y_r} dy$$

Then, we have

(i) $\langle A(t)v,w \rangle = a(t,v,w)$, $\forall$ v,w $\in$ V, where $\langle \ , \ \rangle$ is the duality pairing between V and its topological dual .

(ii) $a(t,v,v) \quad a_0 \|v\|^2$, $\forall$ v $\in$ V, a_0 positive constant

(iii) $|a(t,v,w)| \quad C\|v\| \, \|w\|$, $\forall$ v,w $\in$ V

Proof: The same given in Limaco-Miranda [14].

Lemma 2: If n=3 and let us consider

$= \{v \in L^2(0,T,V) \quad L \ (0,T,H);$

$\int_0^{T-\delta} | v(t+\delta) - v(t) |^2 \, dt \leq C\delta^{1/2}, \forall \quad \in [0,T]\}$

with $\| v \|_\beta = | v |_{L^2(0,T,V)} + | v |_{L^\infty(0,T,H)}$ Then, the embedding $\subset L^p(0,T,L^q(\ ))$ is compact when p $\in$ (2, +), q $\in$ [2,6) and $\frac{1}{p} + \frac{3}{2q} > \frac{3}{4}$

Proof: By Lions [5], Lemma 5.1 pp: 298.

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