American Journal of Applied Sciences 5 (11): 1522-1527, 2008
ISSN 1546-9239

A Novel Fast Computing Method for Framelet Coefficients

Hadeel N. Al-Taai
Department of Electrical and Electronic Engineering,
University of Technology, Baghdad, Iraq

Abstract: The relatively new field of framelets shows promise in removing some of the limitations of wavelets. Several applications have benefited from the use of frames, for example, denoising and signal coding. In this research, Fast 1D-2D Framlet Transform algorithm for computing advance transforms are proposed. For a 2D framelet transformation, the algorithm is applied in x-direction first and then in y-direction. The propose method reduces heavily processing time for decomposition of video sequences keeping or overcoming the quality of reconstructed sequences In addition, it cuts heavily the memory demands. Also, the inverse procedures of all the above transform for multi-dimensional cases are verified.

Key words: Fast computing, framelet transform, filter banks, inverse framelet transform

INTRODUCTION

Though standard DWT is a powerful tool for analysis and processing of many real-world signals and images, it suffers from three major disadvantages, Shift- sensitivity, Poor directionality and Lack of phase information. These disadvantages severely restrict its scope for certain signal and image processing applications[1].

Other extensions of standard DWT such as Wavelet Packet Transform (WP) and Stationary Wavelet Transform (SWT) reduce only the first disadvantage of shift- sensitivity but with the cost of very high redundancy and involved computation. Recent research suggests the possibility of reducing two or more above-mentioned disadvantages using different forms of Wavelet Transforms[2-4] with only limited (and controllable) redundancy and moderate computational complexity.

Frames, or overcomplete expansions, have a variety of attractive features. With frames, better time-frequency localization can be achieved than is possible with bases. Some wavelet frames can be shift invariant, while wavelet bases cannot be. Frames provide more degrees of freedom to carry out design. There are a number of methods of generating practical frames[5]. The undecimated DWT (UDWT) generates a wavelet frame from an existing wavelet basis by removing the subsampling from an existing critically sampled filter bank[6]. A wavelet frame can be obtained by taking the union of two (or more) bases. This can be implemented with two independent filter banks operating in parallel. Kingsbury has shown the advantages of dual-tree DWTs[7].

A wavelet frame can also be obtained by iterating a suitably designed oversampled filter bank as developed in[8], for example. This is the type of frame to be considered in this research.
This research describes new wavelet tight frames based on iterated oversampled FIR filter banks, first introduced in[9]. Selesnick *et al*.[9] introduce the double-density wavelet transform (DDWT) as the tight-frame equivalent of Daubechies' orthonormal wavelet transform; the wavelet filters are of minimal length and satisfy certain important polynomial properties in an oversampled framework. Because the DDWT, at each scale, has twice as many wavelets as the DWT, it achieves lower shift sensitivity than the DWT. New fast computation algorithms for computing discrete framelet transform have been described in this research in a simple and easy to verify procedure based on iterated FIR filter bank that simplify computation complexity by using simple operations like matrix multiplication and addition.

PRELIMINARIES

Framelet are very similar to wavelets but have some important differences. In particular, whereas wavelets have an associated scaling function $\Phi(t)$ and wavelet function $\psi(t)$, framelets have one scaling function $\Phi(t)$ and two wavelet functions $\psi 1(t)$ and $\psi 2(t)$.

The scaling function $\Phi(t)$ and the wavelets $\psi 1(t)$ and $\psi 2(t)$ are defined through these equations by the low-pass (scaling) filter $h_0(n)$ and the two high-pass (wavelet) filters $h_1(n)$ and $h_2(n)$. Let

$$\phi(t) = \sqrt{2}\sum_{n} h_0(n)\ \ \phi(2t-n),$$
$$\psi_i(t) = \sqrt{2}\sum_{n} h_i(n)\ \ \phi(2t-n)\ ,\ i = 1,2. \quad (1)$$

Any function f(t) could be written as a series expansion in terms of the scaling function and wavelets by[5]:

$$f(t) = \sum_{k=-\infty}^{\infty} c(k)\phi_k(t) + \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty} d_1(j,k)\psi_{1,j,k}(t) + d_2(j,k)\psi_{2,j,k}(t) \quad (2)$$

Where

$$c(k) = \int f(t)\phi_k(t)dt$$
$$d_i(j,k) = \int f(t)\psi_{i,j,k}(t)dt \quad i = 1,2 \quad (3)$$

In this expansion, the first summation gives a function that is a low resolution or coarse approximation of f(t) at scale j = 0. For each increasing j in the second summation, a higher or finer resolution function is added, which adds increasing details.

The filters $h_i(n)$ and $h_i(-n)$ should satisfy the perfect reconstruction (PR) conditions. From basic multirate identities, the PR conditions are the following[10]:

$$H_0(z).H_0\left(\frac{1}{z}\right) + H_1(z).H_1\left(\frac{1}{z}\right) + H_2(z).H_2\left(\frac{1}{z}\right) = 2 \quad (4)$$

And

$$H_0(-z).H_0\left(\frac{1}{z}\right) + H_1(-z).H_1\left(\frac{1}{z}\right) + H_2(-z).H_2\left(\frac{1}{z}\right) = 0 \quad (5)$$

Let K_0 denote the number of zeros $H_0(e^{jw})$ has at $w = \pi$. For i = 1,2, let K_i denote the number of zeros $H_i(e^{jw})$ has at w = 0. Then the Z-transform of each $h_i(n)$ factors as follows:

$$H_0(z) = Q_0(z)(z+1)^{K_0} \quad (6)$$

$$H_1(z) = Q_1(z)(z-1)^{K_1} \quad (7)$$

$$H_2(z) = Q_2(z)(z-1)^{K_2} \quad (8)$$

K_0 denotes the degree of polynomials representable by integer translates of $\Phi(t)$ and is related to the smoothness of $\Phi(t)$. K_1 and K_2 denote the number of zero moments of the wavelets filters $h_1(n)$ and $h_2(n)$, provided $K_0>K_1$ and $K_0\geq K_2$. If it is desired for a given class of signals that the wavelets have two zero moments (for example), then the remaining degrees of freedom can be used to achieve a higher degree of smoothness by making K_0 greater than K_1 and K_2. Although the values K_i need not all be equal, there is still the constraint:

$$\text{Length } h_0 \geq K_0 + \min(K_1, K_2) \quad (9)$$

So the minimum length of h_0 is $K_0+\min(K_1, K_2)$. In the orthonormal case $K_0 = K_1$ and $K2 = \infty$ (as $h_2 = 0$), which gives the minimum length of h_0 to be $2K_0$, which is consistent with Daubechies orthonormal filters.

For example, we ask that $K_0 = 5$, $K_1 = K_2 = 2$. It was found that the shortest filters h_0, h_1, h_2 satisfying (4,5) are of length 7, 7 and 5, respectively[5,10]. In this example, there are 4 distinct solutions, one of which is shown in Fig. 1 and shown in Table 1.

Table 1: Set of asymmetric analysis filters (synthesis filters are just the flipped version of these)

$h_0(n)$	$h_1(n)$	$h_2(n)$
0.0762236746486	-0.020547940251	-0.02716023590
0.34908887241859	-0.0941053724585	-0.1243883373
0.60208924236383	-0.122897820901	-0.1301659700
0.44194173824159	0.0613533560838	0.7421378961
0.06082336499856	0 .6063328088167	-0.4604233527
-0.0839238294736	-0.311319898477	0
-0.0320295008244	-0.118815132811	0

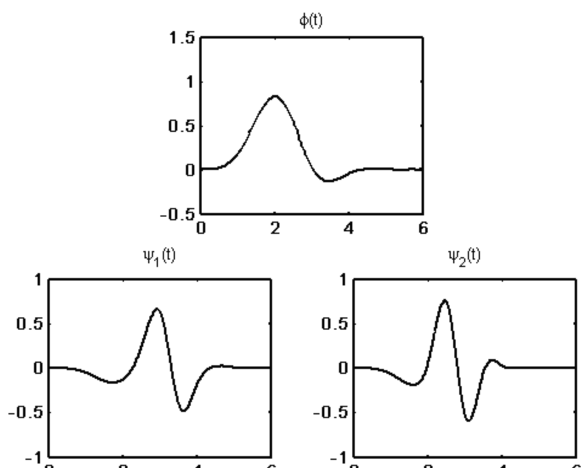

Fig. 1: The generators of a wavelet tight frame with parameters $K_0 = 5$, $K_1 = K_2 = 2$

A PROPOSED FAST COMPUTATION METHOD OF FRAMELET TRANSFORM

1-D Framelet transform: The framelet transform is implemented on discrete-time signals using the over sampled analysis and synthesis filter bank shown in Fig. 2. The analysis filter bank consists of three analysis filters- one low pass filter denoted by $h_0(n)$ and two distinct high pass filters denoted by $h_1(n)$ and $h_2(n)$. As the input signal X(N) travels through the system, the analysis filter bank decomposes it into three sub bands, each of which is then down-sampled by 2. From this process $X_L(N/2)$, $X_{H1}(N/2)$ and $X_{H2}(N/2)$ are generated, which represent the low frequency (or coarse) subband and the two high frequency (or detail) sub bands, respectively.

The up sampled signals are filtered by the corresponding synthesis low pass $h_0^*(n)$ and two high pass $h_1^*(n)$ and $h_2^*(n)$ filters and then added to reconstruct the original signal. Note that the filters in the synthesis stage, are not necessary the same as those in the analysis stage. For an orthogonal filter bank, $h_i^*(n)$ are just the time reversals of $h_i(n)$.

Wavelet frames, having the form described above, have twice as many wavelets than is necessary. Yet note that the filter bank showed in Fig. 2 is oversampled by 3/2, not by 2. However, if the filter bank is iterated a single time on its lowpass branch (h_0), the total oversampling rate will be 7/4. For a three-stage filter bank, the oversampling rate will be 15/8. When this filter bank is iterated on its lowpass branch indefinitely, the total oversampling rate increases toward 2, which is consistent with the redundancy of the frame for $L_2(R)$.

For computing fast discrete framelet transform consider the following transformation matrix for length-7:

$$W=\begin{bmatrix}
h_0(0) & h_0(1) & h_0(2) & h_0(3) & h_0(4) & h_0(5) & h_0(6) & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & h_0(0) & h_0(0) & h_0(2) & h_0(3) & h_0(4) & h_0(5) & h_0(6) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
h_0(2) & h_0(3) & h_0(4) & h_0(5) & h_0(6) & 0 & 0 & 0 & 0 & \cdots & h_0(0) & h_0(1)\\
h_1(0) & h_1(1) & h_1(2) & h_1(3) & h_1(4) & h_1(5) & h_1(6) & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & h_1(0) & h_1(1) & h_1(2) & h_1(3) & h_1(4) & h_1(5) & h_1(6) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
h_1(2) & h_1(3) & h_1(4) & h_1(5) & h_1(6) & 0 & 0 & 0 & 0 & \cdots & h_1(0) & h_1(1)\\
h_2(0) & h_2(1) & h_2(2) & h_2(3) & h_2(4) & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & h_2(0) & h_2(1) & h_2(2) & h_2(3) & h_2(4) & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
h_2(2) & h_2(3) & h_2(4) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & h_2(0) & h_2(1)
\end{bmatrix}_{\frac{3N}{2}\times N} \quad (10)$$

Here blank entries signify zeros and for length-10 become:

$$W=\begin{bmatrix}
h_0(0) & h_0(1) & h_0(2) & h_0(3) & h_0(4) & h_0(5) & h_0(6) & h_0(7) & h_0(8) & h_0(9) & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & h_0(0) & h_0(1) & h_0(2) & h_0(3) & h_0(4) & h_0(5) & h_0(6) & h_0(7) & h_0(8) & h_0(9) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
h_0(2) & h_0(3) & h_0(4) & h_0(5) & h_0(6) & h_0(7) & h_0(8) & h_0(9) & 0 & 0 & 0 & 0 & \cdots & h_0(0) & h_0(1)\\
h_1(0) & h_1(1) & h_1(2) & h_1(3) & h_1(4) & h_1(5) & h_1(6) & h_1(7) & h_1(8) & h_1(9) & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & h_1(0) & h_1(1) & h_1(2) & h_1(3) & h_1(4) & h_1(5) & h_1(6) & h_1(7) & h_1(8) & h_1(9) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
h_1(2) & h_1(3) & h_1(4) & h_1(5) & h_1(6) & h_1(7) & h_1(8) & h_1(9) & 0 & 0 & 0 & 0 & \cdots & h_1(0) & h_1(1)\\
h_2(0) & h_2(1) & h_2(2) & h_2(3) & h_2(4) & h_2(5) & h_2(6) & h_2(7) & h_2(8) & h_2(9) & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & h_2(0) & h_2(1) & h_2(2) & h_2(3) & h_2(4) & h_2(5) & h_2(6) & h_2(7) & h_2(8) & h_2(9) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
h_2(2) & h_2(3) & h_2(4) & h_2(5) & h_2(6) & h_2(7) & h_2(8) & h_2(9) & 0 & 0 & 0 & 0 & \cdots & h_2(0) & h_2(1)
\end{bmatrix}_{\frac{3N}{2}\times N} \quad (11)$$

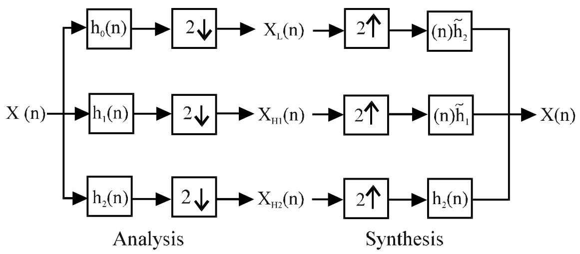

Fig. 2: Analysis and synthesis stages of a 1-D single level discrete framelet transform

To compute a single level for 1-D signal the next steps should be followed:

- Checking input dimensions: Input vector should be of length N, where N must be even and N> = length of analysis filters
- Construct a transformation matrix: using transformation matrices given in (10) and (11)
- Transformation of input vector, which can be done by apply matrix multiplication to the 3N/2xN constructed transformation matrix by the Nx1 input vector

$$Y = [W]_{\frac{3N}{2}\times N}.[X]_{N\times 1}$$

Where:

$$X_L = Y\left[0:\frac{N}{2}-1\right]$$
$$X_{H1} = Y\left[\frac{N}{2}:N-1\right] \qquad (12)$$
$$X_{H2} = Y\left[N:\frac{3N}{2}-1\right]$$

2-D Framelet transform: The choice of the transform to be used depends on a number of factors, in particular, computational complexity and coding gain. Computational complexity is measured in terms of the number of multiplications and additions required for the implementation of the transform. Coding gain is a measure of how well the transformation compacts signal energy into a small number of coefficients.

A 2-D separable transform is equivalent to two 1-D transforms in series. It is implemented as 1-D row transform followed by a 1-D column transform on the data obtained from the row transform. Figure 3 shows the filter bank structure for computation of a 2-D discrete framelet transform.

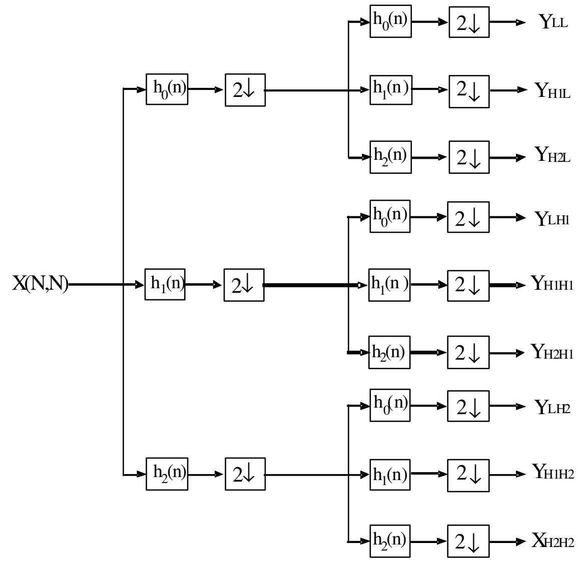

Fig. 3: Filter bank structure for computation of a 2-D discrete framelet transform

framelet transform using nonseparable method, the next steps should be followed:

- Checking input dimensions: Input matrix should be of length NxN, where N must be even and N> = length of analysis filters
- For an NxN matrix input 2-D signal, X, construct a 3N/2xN transformation matrix, W, using transformation matrices given in (10) and (11)
- Apply Transformation by multiplying the transformation matrix by the input matrix by the transpose of the transformation matrix

$$Y = W \cdot X \cdot W^T$$

This multiplication of the three matrices result in the final discrete framelet transformed matrix.

A PROPOSED METHOD OF INVERSE FRAMELET TRANSFORM

1-D Inverse Framelet transform: To reconstruct the original signal from the discrete framelet transformed signal, inverse fast discrete framelet transform should be used. The inverse transformation matrix is the transpose of the transformation matrix as the transform is orthogonal.

To compute a single level 1-D Inverse discrete framelet transform, the following steps should be followed:

- Let Y be the 3N/2x1 framelet transformed vector.
- Construct Nx3N/2 reconstruction matrix, $T = W^t$, using transformation matrices given in (10) and (11)
- Reconstruction of input vector, which can be done by applying matrix multiplication to the Nx3N/2 reconstruction matrix, T, by the 3N/2x1 framelet transformed vector

$$X = [T]_{N\times\frac{3N}{2}} . [Y]_{\frac{3N}{2}\times 1}$$

2-D Inverse Framelet transform: To compute a single level inverse framelet transform for 2-D signal using non-separable method the next steps should be followed:

- Let Y_o be the 3N/2x3N/2 framelet transformed matrix.
- Construct Nx3N/2 reconstruction matrix, $T = W^t$, using transformation matrices given in (10) and (11)
- Reconstruction of the input matrix by multiplying the reconstruction matrix by the input matrix by the transpose of the reconstruction matrix. $X = T \cdot Y_o \cdot T^t$

A COMPUTER TEST

During a single level of decomposition using wavelet transform, the 2-D image data is replaced with four blocks corresponding to the subbands representing either lowpass or highpass in both dimensions. These sub bands are shown in Fig. 4a. The subband labels in this figure indicate how the subband data is generated. For example, the data in subband LH is obtained from highpass filtering of the rows and then low pass filtering of the columns[6]. The framelet transform is extended to 2D by iterating the 1D over sampled filter bank on the rows and then on the columns of an image, as is usually done for separable 2D- framelet transform. At a given level in the iterated filter bank, this separable extension produces nine 2D subbands. These subbands are showed in Fig. 4b. Since L is a lowpass filter ($h_0(n)$) while both H_1 and H_2 are highpass filters ($h_1(n)$ and $h_2(n)$), the H_2H_2, H_2H_1, H_1H_2, H_1H_1 subbands each have a frequency-domain support comparable to that of the HH subband in a DWT. A similar scheme creates the H_1L, H_2L (LH_1, LH_2) subband with the same frequency-domain support as the corresponding HL (LH) subband of the DWT, but with twice as many coefficients. Finally, note that there is only one subband LL with the same frequency-domain support as the LL subband in a DWT.

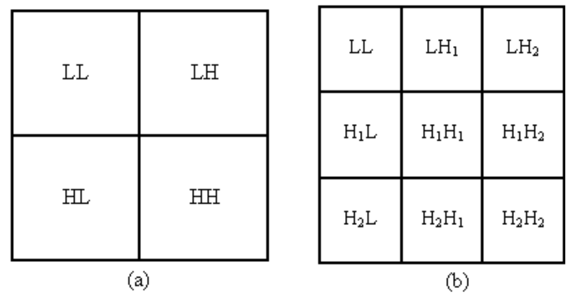

Fig. 4: Image Subbands after a single-level decomposition, for (a): Scalar Wavelets and (b): Framelets

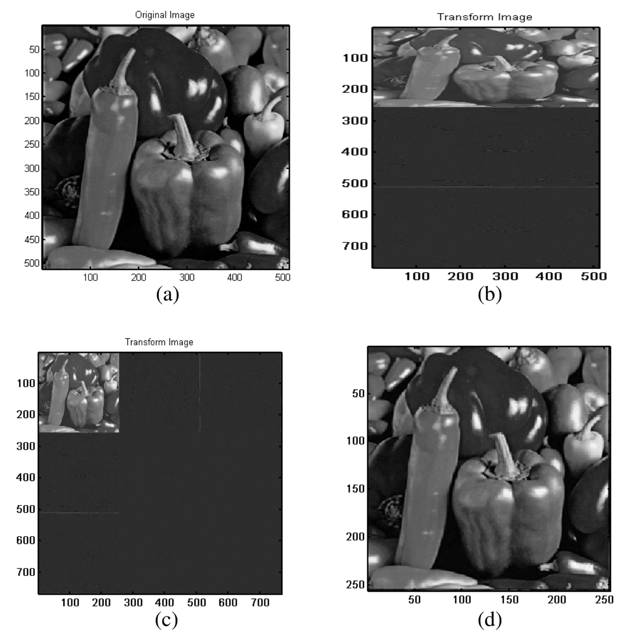

Fig. 5: Peppers image, (a): Original, (b): After input row transform, (c): After input column transform, (d): The upper-left most, LL, subband

A general computer program computing a single-level fast discrete framelet transform is written using Matlab V.7.0 for a general NxN 2-D signal (or image). As shown in Fig. 5a, the original Peppers image dimensions are 512x512 (NxN). After a single-level of framelet decomposition using seperable method, after row transformation image dimensions will be a matrix of 768x512 (3N/2xN), as shown in Fig. (5b), after the column transformation , image dimensions will be a matrix of 768x768 (3N/2x3N/2), as shown in Fig. 5c. The upper-left most, LL, subband of 256x256 dimensions, is zoomed in as in Fig. 5d.

CONCLUSIONS

This research presents a new framelets transform computation methods from 1D-2D that verify the potential benefits of framelets and gain a much improvement in terms of low computational complexity. The relatively new field of framelets shows promise in removing some of the limitations of wavelets. Several applications have benefited from the use of frames, for example, denoising and signal coding.

REFERENCES

1. Shukla, P.D., 2003. Complex wavelet transform and their applications. M.Phil. Thesis, Department of Electronic and Electrical Engineering, University of Strathclyde, Scotland.
2. Fernandes, F., 2002. Directional, shift-insensitive, complex wavelet transforms with controllable redundancy. Ph.D Thesis, Rice University.
3. Spaendonck, R. *et al*., 2003. Orthogonal hilbert transform filterbanks and wavelets. ICASSP03, April 6-10.
4. Fernandes, F. *et al*., 2000. Directional complex wavelet processing. In Wavelet Applications VII, Proceeding of SPIE.
5. Selesnick, I.W., 2001. Smooth wavelet tight frames with zero moments. Applied and Computational Harmonic Analysis (ACHA), 10 (2): 163-181.
6. Rioul, O. and P. Duhamel, 1992. Fast algorithms for discrete and continuous wavelet transforms. IEEE Trans. Inform. Theory, 38: 569-586.
7. Kingsbury, N.G., 1998. The Dual-Tree Complex Wavelet Transform: A New Technique for Shift Invariance and Directional Filters. Proceedings of the Eighth IEEE DSP Workshop, Utah, August 9-12.
8. Chui, C. and W. He, 2000. Compactly supported tight frames associated with refinable functions. Applied Comput. Harmon. Anal., 8: 293-319.
9. Selesnick, I.W. and L. Sendur, 2000. Iterated oversampled filter banks and wavelet frames. In: Wavelet Applications in Signal and Image Processing VIII, San Diego. Proceedings of SPIE, Vol. 4119.
10. Selesnick, I.W., 2004. The double-density dual-tree DWT. IEEE Trans. Signal Processing, 52 (5): 1304-1314.