Original Research Paper

Level of Non-Matching Primary Transplant Cases According to Repeat Transplants Cases and a New Bivariate Poisson Distribution

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Abstract: In their practice, healthcare administrators and professionals often wonder about the non-matching level organs in transplants for the sake of future forecasting. Currently, there is no appropriate methodology to analyze the pertinent transplant data and describe the patterns. The lack of a suitable methodology in the literature originates from an incorrect impression that the primary transplant cases and the repeat transplant cases are two separate and independent Poisson probability processes. In fact, the actual data on the primary and repeat transplant cases in USA during the year 2014 indicate otherwise with a high degree of correlation between them. One wonders about the missing link and it hides in their model as this article articulates. The aims of this article are set to find an appropriate underlying model for the data and then construct an analytic methodology. In this research process, a novel and useful bivariate probability distribution is discovered and it is named here *"seemingly independent bivariate Poisson distribution"* for a lack of better title. Its statistical properties are derived, explained and illustrated. This new bivariate distribution helps not only to estimate the non-matching level of organs in the transplant cases but also to project the number of repeat transplant cases based on knowing the number of primary transplant cases and vice versa.

Keywords: Correlation, Regression, Conditional Mean and Variance, Risk

Introduction

Motivation

Organ transplantation is too important medical procedure to miss not learning enough details (Bentley, 2014 for details). Transplanting organs or tissues within the same person's body are auto-grafts. Transplants occurring between two different persons are allografts. In each scenario, non-matching transplants could occur. Tydén *et al*. (2012) and Urschel *et al*. (2013) for details on how the blood group incompatibility results in non-matching and failure in transplanted hearts. Estimating the level of non-matching is a precursor to improve the transplant process (Almond *et al*., 2010). Klymiuk *et al*. (2010) to learn about how the genetics play a crucial role in detecting non-matches of organs. The organs that have been successfully transplanted so far include hearts, kidneys, livers, lungs, pancreas, intestines and thymus. Some organs, like the brain, are not transplanted. Tissues like the bones, tendons (both referred to as musculoskeletal grafts), cornea, skin, heart valves, nerves and veins are transplantable. Most commonly transplanted organs are kidneys, followed by livers and then hearts (Gupta *et al*., 2015 for data). Failure in primary transplanted cases proceed to repeat transplant cases. Naesens (2014) for historical reasons for the failure of transplanted kidneys. Saczkowski *et al*. (2010) for reasons of the failure in the transplanted hearts. Cornea and musculoskeletal grafts are most commonly repeat-transplanted tissues. Shanmugam, 2015b) for their incidences.

How important is then the topic of transplants? For example, as of December 31, 2013, about 86,965 candidates wait for kidney transplant and it is rather a high number. Only about 83% of those candidates were awaiting for their first transplant and the remaining 17%

waited for their repeat transplant. It is easy to realize the gravity of repeat transplant based on the numbers. Among the new candidates wait-listed for either a first time or a repeat kidney-alone transplant in 2009, the median waiting time to transplant was about 3.6 years and it suggests a great need to improve the system. Any improvement is not feasible until we learn and comprehend the information in the data. Patient survival is considerably better following repeat transplant, which is not surprising given that patients receiving second transplants are on the average younger (West, 2011). Shanmugam (2013a) for details about a technique how to estimate the chance for more survival time if a cancerous kidney is removed. Vera *et al*. (2009) for non-matching issues with respect to the liver transplants. In addition, there are ethical issues with respect to repeat transplants (Dobbels *et al*., 2012). Len *et al*. (2014) advocate and encourage the importance of screening and selecting organs for transplant. Whether it was for primary or repeat scenario, the transplants (Daniels, 2008; Magee *et al*., 2007; Piccoli *et al*., 2004) helped to extend the survival time. In this globalized economy, the patients in need of transplants do make a medical tour to another country and get less expensive organs transplanted (Budiani-Saberi and Delmonico, 2008).

The data for analyses and interpretations in this article are downloaded from the public domain http://optn.transplant.hrsa.gov. Motivated by the data on the *primary* and *repeat* transplant cases in USA (Table 1-4 in Eastern, Central, Mountain and Pacific regions

respectively) in the year 2014, this article is prepared to guide the data analysts on how to extract and interpret pertinent data evidence for the increased survival time. Notice that the primary and repeat transplant cases are highly correlated whether the data pertain to the Eastern, Central, Mountain, or Pacific region in USA, though individually primary as well as repeat transplant cases are seemingly follow two separate and independent Poisson probability distributions with different incidence rates. The Poisson distribution is a versatile model to capture and explain how the chance mechanism adapts to adverse rare outcomes (Shanmugam, 2013b for details).

After searching the literature, we realized that there is no appropriate probability model to fit and explain together both primary and repeat incidences data trend. Hence, we introduce a novel bivariate probability distribution and name it as "*Seemingly Independent Bivariate Poisson (SIBP) distribution*". This new bivariate distribution helps to predict the number of repeat transplant cases based on knowing the number of primary transplant cases and vice versa. In general, bivariate probability distributions play a key role in both the analyses and interpretations of medical or health data. Shanmugam (2015a; 2014a; 2014b; 2014c) for details about the vital role of bivariate probability distribution in health data analyses. Considering hacking versus vigilance as opposite but two different uncertain events, Shanmugam (2013c) recently developed and demonstrated a new bivariate probability distribution to assess the cyber insecurity level.

Table 1. Primary and repeat organ transplant cases in the Eastern US region during 2014

States	Y	X	$\hat{X}$	$\hat{Y}$	Volatility of Y	Volatility of X
Alabama	14,319	1,634	1191.62	14122.46	4.58024E-06	0.003082
Connecticut	2,845	404	299.96	10383.10	1.15295E-06	0.002266
DC	13,324	1,699	1238.75	13798.19	4.76136E-06	0.003011
Florida	39,029	4,866	3534.61	22175.41	1.35859E-05	0.004839
Georgia	16,607	1,799	1311.24	14868.11	5.04E-06	0.003244
Indiana	11,844	1,304	952.40	13315.86	3.66072E-06	0.002906
Kentucky	7,222	731	537.01	11809.55	2.06411E-06	0.002577
Maryland	18,898	2,755	2004.28	15614.75	7.70381E-06	0.003407
Massachusetts	26,879	2,865	2084.02	18215.74	8.01032E-06	0.003975
Michigan	22,489	2,939	2137.66	16785.05	8.21651E-06	0.003663
New Jersey	17,378	2,230	1623.69	15119.38	6.24094E-06	0.003299
New York	53,672	6,730	4885.89	26947.54	1.87798E-05	0.00588
North Carolina	22,220	2,414	1757.07	16697.38	6.75364E-06	0.003643
Ohio	27,243	3,557	2585.67	18334.37	9.93852E-06	0.004001
Pennsylvania	61,171	8,271	6003.01	29391.46	2.30737E-05	0.006413
Puerto Rico	2,618	185	141.19	10309.12	5.42726E-07	0.002249
South Carolina	6,289	883	647.20	11505.49	2.48764E-06	0.002511
Virginia	14,563	1,552	1132.18	14201.98	4.35176E-06	0.003099
$\hat{\lambda}_2 = \bar{x}$		2,601				
$\hat{\lambda}_1 = \bar{y} - \bar{x}$	18,433					
$\hat{\phi} \approx r\sqrt{\dfrac{\bar{y} - \bar{x}}{\bar{x}}}$	2.63					

Table 2. Primary and repeat organ transplant cases in Central US region during 2014

States	Y	X	$\hat{X}$	$\hat{Y}$	Volatility of Y	Volatility of X
Arkansas	3,356	367	269.79	8269.91	1.02491E-06	0.002361
Illinois	33,477	4,597	3300.84	18006.63	1.25392E-05	0.00514
Iowa	4,508	703	510.56	8642.30	1.93952E-06	0.002467
Kansas	7,611	1,113	804.35	9645.35	3.05556E-06	0.002753
Louisiana	13,275	1,464	1055.86	11476.26	4.01101E-06	0.003276
Minnesota	21,156	3,751	2694.63	14023.83	1.02364E-05	0.004003
Mississippi	1,889	198	148.69	7795.69	5.64878E-07	0.002225
Missouri	13,628	1,369	987.79	11590.37	3.75241E-06	0.003308
Nebraska	6,317	1,075	777.12	9227.06	2.95212E-06	0.002634
Oklahoma	6,482	729	529.19	9280.40	2.01029E-06	0.002649
Tennessee	16,237	2,033	1463.58	12433.74	5.55986E-06	0.003549
Texas	61,445	6,516	4675.92	27047.39	1.77629E-05	0.00772
Wisconsin	17,673	2,560	1841.21	12897.93	6.99438E-06	0.003681
$\hat{\lambda}_2 = \bar{x}$		2,037				
$\lambda_1 = \bar{y} - \bar{x}$	15,927					
$\hat{\phi} \approx r\sqrt{\frac{\bar{y}-\bar{x}}{\bar{x}}}$	2.53					

Table 3. Primary and repeat organ transplant cases in Mountain US region during 2014

States	Y	X	$\hat{X}$	$\hat{Y}$	Volatility of Y	Volatility of X
Arizona	12,630	1,272	933.05	6713.96	3.59E-05	0.025237
Colorado	10,907	1,247	914.86	6173.87	3.52E-05	0.023207
Nevada	1,638	224	170.50	3268.42	6.56E-06	0.012286
New Mexico	1,960	233	177.05	3369.35	6.81E-06	0.012665
Utah	4,840	776	572.15	4272.11	2.2E-05	0.016059
$\hat{\lambda}_2 = \bar{x}$		750				
$\lambda_1 = \bar{y} - \bar{x}$	6,395					
$\hat{\phi} \approx r\sqrt{\frac{\bar{y}-\bar{x}}{\bar{x}}}$	2.67					

Table 4. Primary and repeat organ transplant cases in Pacific US region during 2014

States	Y	X	$\hat{X}$	$\hat{Y}$	Volatility of Y	Volatility of X
California	108,697	12,069	8900.09	44808.47	4.12879E-06	0.001686
Hawaii	1,953	113	91.24	11586.91	4.23307E-08	0.000436
Nevada	1,638	224	173.03	11488.88	8.02696E-08	0.000432
Oregon	5,477	654	489.84	12683.67	2.2724E-07	0.000477
Washington	12,190	1,390	1032.10	14772.94	4.78799E-07	0.000556
$\hat{\lambda}_2 = \bar{x}$		2,890				
$\lambda_1 = \bar{y} - \bar{x}$	25,991					
$\hat{\phi} \approx r\sqrt{\frac{\bar{y}-\bar{x}}{\bar{x}}}$	2.80					

In section 2, various statistical properties of our new *SIBP* distribution including their correlation, marginal, conditional probability structures are derived and then they are utilized to capture non-obvious intricacies between the primary and repeat transplant cases. One of the intricacies is an estimate of non-matching in the primary transplant cases. In section 3, the analytic expressions are illustrated using the data on total primary versus repeat transplant cases in USA (Table 1-4) in the year 2014. Finally, in section 4, a few final comments and conclusions are compiled and stated.

Seemingly Independent Bivariate Poisson Distribution and its Statistical Properties

Let Y and X be two dependent (with a *correlation*, $-1 \leq \rho \leq 1$) Poisson distributed random variables with *incidence rates* $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively. Let $\phi > 0$ be a *bonding parameter*, which can break down their dependence or glue together stochastically the two random variables. In the context of transplant cases, the *bonding parameter* portrays the *non-matching level* of the organ transplants. Then, we propose a

novel *seemingly independent bivariate Poisson (SIBP)* distribution (1) below to model for predictability of the number of repeat transplant cases based on knowing the number of primary transplant cases and vice versa. That is:

$$p\left(Y=y, X=x \middle| \lambda_1, \lambda_2, \phi\right)$$

$$= \begin{cases} \dfrac{\left[1+\phi\dfrac{y!}{(y-x)!}\left(\dfrac{1}{\lambda_1}\right)^x\right]e^{-(\lambda_1+\lambda_2)}\lambda_1^y\lambda_2^x}{(1+\phi)y!x!}; for \begin{cases} x \le y; \\ x > y; \end{cases} \\ \qquad\qquad 0 \end{cases} \quad (1)$$

$$y = 0,1,2,.....,\infty; 0 < \lambda_1, \lambda_2, \phi < \infty.$$

It is obvious that $p(X=x, Y=y|\lambda_1, \lambda_2, \phi) > 0$. After algebraic simplifications, we can show that $\sum_{y=0}^{\infty}\sum_{x=0}^{y} p\left(Y=y, X=x \middle| \lambda_1, \lambda_2, \phi\right) = 1$. Hence, the expression (1) is indeed a bona fide probability function. A specific probability:

$$p\left(Y=0, X=0 \middle| \lambda_1, \lambda_2, \phi\right)$$

in (1) of no primary and no repeat transplant incidence portrays the likelihood of a healthy community. The odds for a *healthy community* to exist is displayed in Fig. 1, where its odds are identified in z-axis, the primary

transplant incidence rate, λ_2 in x-axis and the repeat incidence rate λ_1 in y-axis). That is:

$$Odds_{(0,0)}$$

$$= \frac{p\left(Y=0, X=0 \middle| \lambda_1, \lambda_2, \phi\right)}{1 - p\left(Y=0, X=0 \middle| \lambda_1, \lambda_2, \phi\right)} \quad (2)$$

$$= \left(e^{\lambda_1+\lambda_2} - 1\right)^{-1}$$

The $Odds_{(0,0)}$ in (2) for having a healthy community diminishes as the incidence rate λ_1 for the primary or the incidence rate λ_2 for the repeat transplant increases from a zero level. Likewise, the specific probability:

$$p\left(Y=1, X=0 \middle| \lambda_1, \lambda_2, \phi\right)$$

of one primary but no repeat transplant portrays a scenario of an *efficient community* (because no need for repeat transplant). The odds for such an efficient is:

$$Odds_{(1,0)}$$

$$= \frac{p\left(Y=1, X=0 \middle| \lambda_1, \lambda_2, \phi\right)}{1 - p\left(Y=1, X=0 \middle| \lambda_1, \lambda_2, \phi\right)} \quad (3)$$

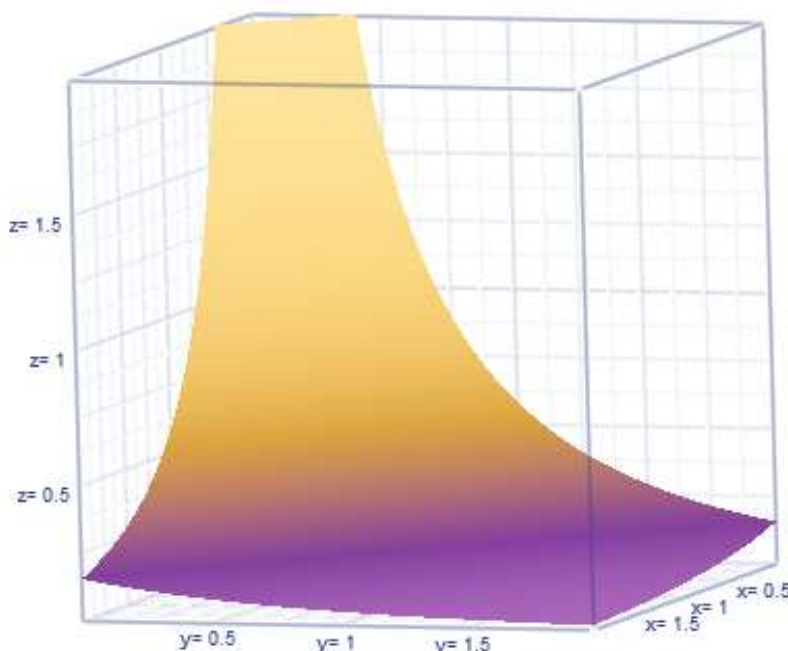$$= \lambda_1\left(e^{\lambda_1+\lambda_2} - \lambda_1\right)^{-1}$$

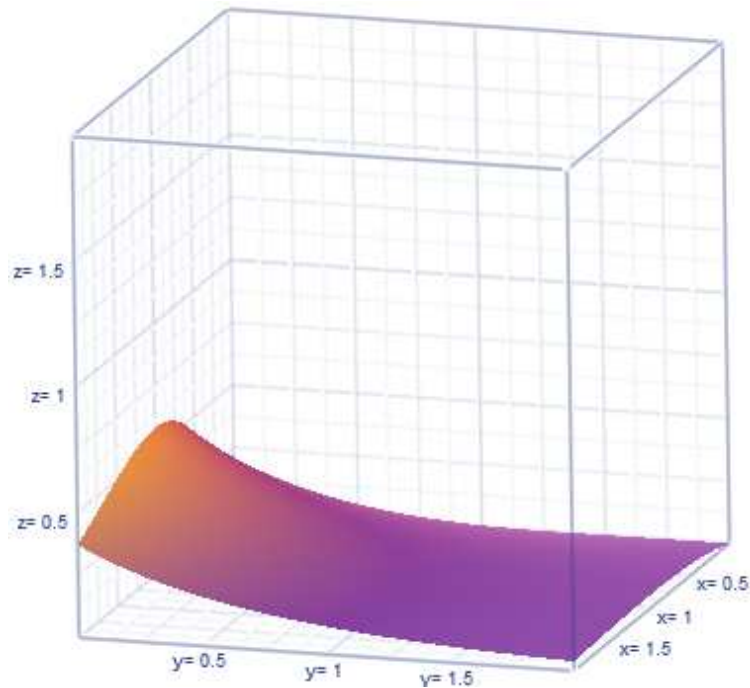Fig. 1. Odds for healthy community

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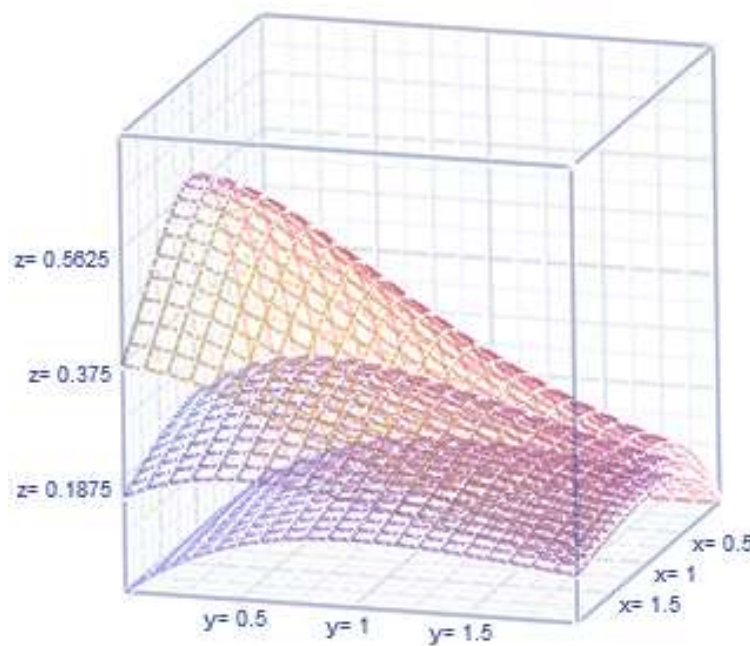Fig. 2. Odds for efficient community

Fig. 3. Odds for inefficient community

The odds for an efficient community to exist is displayed in Fig. 2. The $Odds_{(1,0)}$ starts at a finite level, depending on just the primary incidence rate λ_1 but concavely (Fig. 2) diminishes.

However, the probability:

$$p\left(Y=1, X=1 \middle| \lambda_1, \lambda_2, \phi\right)$$

Portrays the likelihood of having one primary transplant and a need to have another repeat transplant (because either the primary transplant was done poorly

or the patient's body rejects the non-matching transplanted organ or both). This scenario amounts to having an *inefficient primary transplant*. The odds (Fig. 3 with $\phi = 0, 0.5, 1$) of having *inefficient primary transplant* is therefore:

$$Odds_{(1,1)}$$
$$= \frac{p(Y=1, X=1|\lambda_1, \lambda_2, \phi)}{1 - p(Y=1, X=1|\lambda_1, \lambda_2, \phi)} \qquad (4)$$
$$= \frac{\lambda_2(\lambda_1 + \phi)}{\left[e^{\lambda_2 + \lambda_1} - \lambda_2(\lambda_1 + \phi) \right]}$$

where, $\phi \geq 0$ is recognized as the level of *nonmatching primary transplant*. In other words, with $\phi = 0$, the scenario refers the absence of non-matching primary transplant. This scenario would not have a need for repeat transplant. Only in this scenario, the *SIBP* probability mass function (1) reduces to:

$$p(Y=y, X=x|\lambda_1, \lambda_2, \phi=0)$$
$$= \frac{e^{-(\lambda_1 + \lambda_2)} \lambda_1^y \lambda_2^x}{y! x!}$$
$$= p(Y=y|\lambda_1) p(X=x|\lambda_2)$$

It implies then that only this scenario portrays the stochastic independence between the primary and repeat transplant incidences. In a statistical sense, the bonding parameter with an attainment of a particular value $\phi = 0$ breaks down the intricate relation between the primary and the repeat transplant Poisson incidences. Of course, by definition or conceptually viewing, the probability,

$$p(Y=0, X=1|\lambda_1, \lambda_2, \phi) = 0$$

of *no primary but a repeat transplant* is impossible. Hence, this scenario is null and void with a thought that the $Odds_{(0,1)} = 0$ in a sense of degeneration.

Now, let us proceed to examine how does the number of primary transplants marginally behave without a knowledge about the number of repeat transplant cases. That is:

$$p(Y=y|\lambda_1, \lambda_2, \phi)$$
$$= \sum_{x=0}^{y} p(Y=y, X=x|\lambda_1, \lambda_2, \phi)$$
$$= \frac{\left[1 + \phi e^{-\lambda_2} \left(1 + \frac{\lambda_2}{\lambda_1} \right)^y \right]}{(1+\phi)} \frac{e^{-\lambda_1} \lambda_1^y}{y!}; \qquad (5)$$
$$y = 0, 1, 2, \ldots, \infty; 0 < \lambda_1, \lambda_2, \phi < \infty.$$

which is nothing but a *size-biased* version of the regular Poisson probability mass function. The size-biased sampling has been studied by statisticians as the sampled population is quite different from the intended population in a sampling process (Shanmugam and Singh, 2012 along with the cited references there). When there is an absolute *absence of non-matching* transplant cases (that is, $\phi = 0$), the number of primary transplant cases is really a separate Poisson incidences with a rate $\lambda_1 > 0$ (that is, $p(Y=y|\lambda_1, \lambda_2, \phi) \to \frac{e^{-\lambda_1} \lambda_1^y}{y!}$ as $\phi \to 0$). The marginal expected value and variance of the primary transplants in that scenario are then:

$$E(Y=y|\lambda_1, \lambda_2, \phi)$$
$$= \lambda_1 + \frac{\phi}{(1+\phi)} \lambda_2 \qquad (6)$$

and:

$$Var(Y=y|\lambda_1, \lambda_2, \phi)$$
$$\approx (1+\phi)\lambda_1 + \phi\lambda_2(\lambda_2 + 2) \qquad (7)$$

From the expression (7), we realize that the expected number of the primary transplants has an add-on to what it would have been when there is an absence of nonmatching transplant (that is, $\phi = 0$). Such an add-on number is $\frac{\phi}{(1+\phi)} \lambda_2$ which is controlled by not only the non-matching level ϕ but also the repeat transplant rate λ_2. As mentioned in the motivation, an intricacy between the primary transplant incidences and the repeat transplant incidences is established in clear terms. The incorrect impression of seemingly unrelated occurrences of primary and repeat transplant occurrences is set correct now by the *SIBP* distribution (1).

Likewise, in the presence of non-matching transplant cases (that is, $\phi \neq 0$), the variance (7) increases from λ_1 (which is what it would have been in the absence of non-matching primary transplant cases) with an add-on amount $\phi[\lambda_1 + \lambda_2 (\lambda_2 + 2)]$. This add-on amount suggests that the number of primary transplants is more volatile (Fig. 4) due to a higher non-matching level (that is, $\phi \neq 0$) in primary transplant cases. The volatility is further controlled by a higher primary transplant occurrence rate λ_1 and a higher rate λ_2 of repeat transplant cases. This is another important intricate relation between the primary and repeat transplant cases and it would not have been possible without our new *SIBP* distribution (1).

Furthermore, only when $\phi = 0$, the variance (7) is exactly equal to the expected value (6), validating the unique characteristics property of two separate Poisson incidences. The functional relationship between the

variance and the mean provides unexpected clues in the data to characterize the Poisson chance mechanism (see Shanmugam, 2014d) for details). The Fig. 5 depicts the nonlinear behavior of the variance (7).
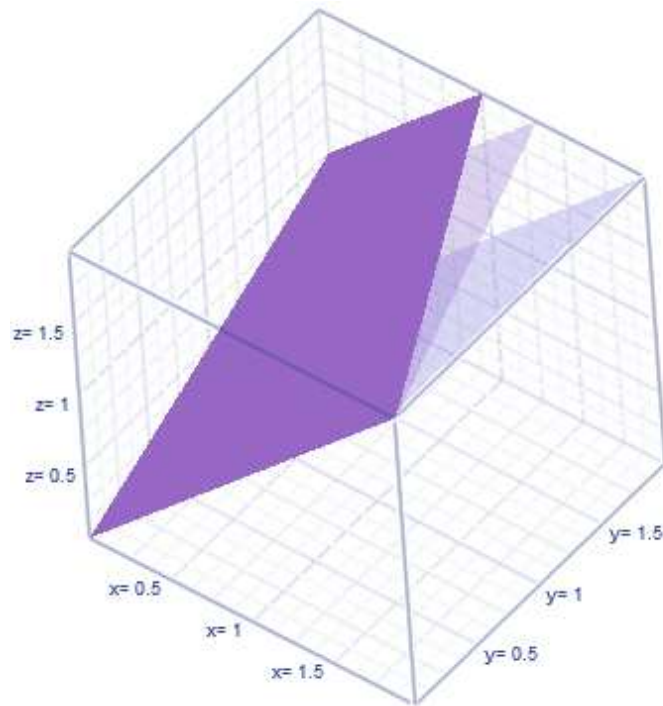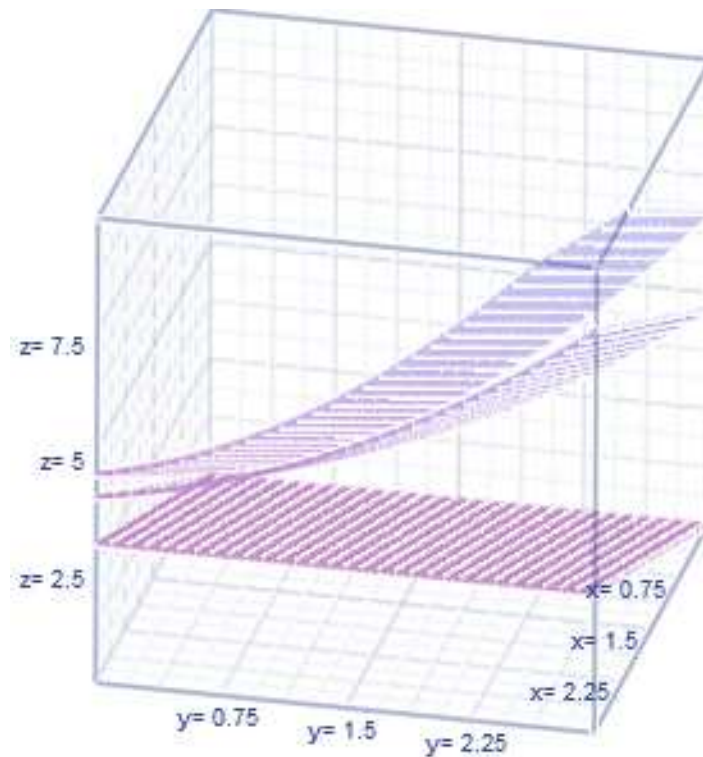

Fig. 4. Mean of SIBP with $\phi = 0, 0.5, 1$

Fig. 5. Variance of SIBP with $\phi = 0, 0.5, 1$

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Hence, the conditional probability mass function, $p(X = x|Y = y,\lambda_1,\lambda_2,\phi)$ of the number of repeat transplant cases for a given marginal number $Y = y|\lambda_1,\lambda_2,\phi$ of primary transplants is estimable. That is:

$$p\left(X = x|Y = y,\lambda_1,\lambda_2,\phi\right)$$
$$= \frac{p\left(Y = y, X = x|\lambda_1,\lambda_2,\phi\right)}{p\left(Y = y|\lambda_1,\lambda_2,\phi\right)}$$
$$= \frac{\left[1 + \dfrac{\phi y!}{(y-x)!\lambda_1^x}\right]}{\left[1 + \phi e^{-\lambda_2}\left(1 + \dfrac{\lambda_2}{\lambda_1}\right)^y\right]} \frac{e^{-\lambda_2}\lambda_2^x}{x!}; \quad (8.1)$$
$$x = 0,1,2,.....,y; 0 < \lambda_1,\lambda_2,\phi < \infty.$$

Notice that the conditional probability mass function (8) is also size-biased, but the bias vanishes in the absence of nonmatching transplant (that is, when $\phi = 0$). In an event that there is just one primary transplant, how likely it matches so that there will be no repeat transplant? Such a probability is:

$$p\left(X = 0|Y = 1,\lambda_1,\lambda_2,\phi\right)$$
$$= \frac{(1+\phi)\lambda_1 e^{-\lambda_2}}{\left(1 + \phi e^{-\lambda_2}\right)\lambda_1 + \phi e^{-\lambda_2}\lambda_2} \quad (8.2)$$
$$\rightarrow e^{-\lambda_2}, if \phi \rightarrow 0$$

Furthermore, the projectile, $E(X = x|Y = y,\lambda_1,\lambda_2,\phi)$ of the number of repeat transplant cases for a given number $Y = y|\lambda_1,\lambda_2,\phi$ of primary transplant cases is:

$$E\left(X = x|Y = y,\lambda_1,\lambda_2,\phi\right)$$
$$\approx \lambda_2\left[1 + \phi\left\{\frac{y}{\lambda_2 + \lambda_1} + 1\right\}\right] \quad (9)$$

The projectile (9) would have been the rate λ_2 of the repeat transplant occurrences only in the absence of non-matching scenario (that is, when $\phi = 0$). The expression (9) is indeed a nonlinear regression for $X = x$ in terms of $Y = y,\lambda_1,\lambda_2,\phi$. The conditional variance (describing the volatility of the number x of repeat transplants) is then:

$$Var(X = x|Y = y,\lambda_1,\lambda_2,\phi)$$
$$\approx \lambda_2\left[1 + \phi\left(\frac{\lambda_2}{\lambda_2 + \lambda_1}\right)y(y-1)\left(1 + \frac{\lambda_2}{\lambda_1}\right)^y\right] \quad (10)$$

The conditional variance (10) is also nonlinear and it converges to the rate λ_2 of the repeat transplant occurrences only in the absence of no matching (that is,

when $\phi = 0$). The above intricate relations between the primary and repeat transplant cases would have been missed without the *SIBP* distribution (1).

Analogously, let us proceed to examine how the number of repeat transplant cases behave marginally. That is:

$$p\left(X = x|\lambda_1,\lambda_2,\phi\right)$$
$$= \sum_{y=0}^{\infty} p\left(Y = y, X = x|\lambda_1,\lambda_2,\phi\right) \quad (11)$$
$$= \frac{e^{-\lambda_2}\lambda_2^x}{x!};$$
$$x = 0,1,2,.....,\infty; 0 < \lambda_2.$$

which is, interestingly, the regular Poisson probability mass function with incidence rate λ_2 of the repeat transplant cases. In a marginal sense, the number of repeat transplant cases do not control the chance for non-matching of the number of primary transplant cases. There exists asymmetry in the intricate relations between the primary and repeat transplant cases. In practical terms, the asymmetry portrays traceability. Such an asymmetry would have been undiscovered without the *SIBP* distribution (1).

The marginal expected value and variance of repeat transplants are then:

$$E\left(X = x|\lambda_2\right) = \lambda_2 \quad (12)$$

and:

$$Var\left(X = x|\lambda_2\right) = \lambda_2 \quad (13)$$

Hence, the conditional probability mass function, $p(Y = y|X = x,\lambda_1,\lambda_2,\phi)$ of the number of primary transplant cases for a given marginal number $X = x,\lambda_1,\lambda_2,\phi$ of repeat transplant cases is tractable. That is:

$$p\left(Y = y|X = x,\lambda_1,\lambda_2,\phi\right)$$
$$= \frac{p\left(Y = y, X = x|\lambda_1,\lambda_2,\phi\right)}{p\left(X = x|\lambda_1,\lambda_2,\phi\right)}$$
$$= \frac{\left[1 + \dfrac{\phi y!}{(y-x)!\lambda_1^x}\right]}{[1 + \phi]}\frac{e^{-\lambda_1}\lambda_1^y}{y!}; \quad (14.1)$$
$$y = x, x+1,.....,\infty; 0 < \lambda_1,\lambda_2,\phi < \infty$$

which is also a size-biased Poisson distribution with incidence rate λ_1. However, the size-bias, $\dfrac{\left[1 + \dfrac{\phi y!}{(y-x)!\lambda_1^x}\right]}{[1+\phi]}$ vanishes in an absence of non-matching

transplant cases (that is, when $\phi = 0$). In an event that there is just one repeat transplant, how likely there might had been two or more primary transplant cases. Such probability is:

$$
p\left(Y \geq 2 \middle| X = 1, \lambda_1, \lambda_2, \phi\right)
$$
$$
= 1 - \frac{[\lambda_1 + \phi]e^{-\lambda_1}}{[1 + \phi]} \tag{14.2}
$$

Furthermore, the projectile, $E(Y = y | X = x, \lambda_1, \lambda_2, \phi)$ of the number of primary transplant cases for a given number $X = x | \lambda_1, \lambda_2, \phi$ of repeat transplant cases is:

$$
E\left(Y = y \middle| X = x, \lambda_1, \lambda_2, \phi\right)
$$
$$
= \lambda_1 + \left(\frac{\phi}{1 + \phi}\right)x \tag{15}
$$

The expression (15) is indeed a linear regression, in terms of the repeat transplant cases $X = x$, with incidence rate λ_1 as intercept and the proportion $\left(\dfrac{\phi}{1+\phi}\right)$ as its slope. The conditional variance (describing the volatility of the number y of primary transplant cases) is then:

$$
Var\left(Y = y \middle| X = x, \lambda_1, \lambda_2, \phi\right)
$$
$$
\approx \left[1 + \phi x^2\right]\lambda_1 \tag{16}
$$

The variance (16) is also nonlinear but it converges to the rate λ_1 of independent primary transplant occurrences, in the absence of non-matching transplant cases (that is, $\phi = 0$). Otherwise (that is, in the presence of non-matching transplant cases indicated by $\phi \neq 0$), the volatility is more with an add-on amount $\phi x^2 \lambda_1$ which is controlled by the non-matching level ϕ, the occurrence rate of primary transplant cases λ_1 and the number x of repeat transplant cases. These are extra intricate relations and revelations that would have been missed without the *SIBP* distribution (1).

Now, we proceed to assess the correlation between the primary and repeat transplant occurrences. For this purpose, using *seemingly independent bivariate Poisson (SIBP)* distribution (1), we first obtain the product moment $E(xy)$. That is:

$$
E(xy)
$$
$$
= \sum_{y=0}^{\infty}\sum_{x=0}^{y} xyp\left(Y = y, X = x \middle| \lambda_1, \lambda_2, \phi\right) \tag{17}
$$
$$
= \lambda_2\left[\lambda_1 + \frac{\phi(\lambda_2 + 1)}{(1 + \phi)}\right]
$$

Substituting the expression (17) in $Cov(x,y) = E(xy) - E(x)Ey$), we obtain that $Cov(x, y) = \dfrac{\phi\lambda_2}{(1 + \phi)}$. Using (7) and (13), we obtain the correlation (18) between the primary and repeat transplant cases. That is:

$$
Corr(x, y) = \frac{Cov(x, y)}{\sqrt{Var(x)Var(y)}}
$$
$$
= \frac{\phi\sqrt{\lambda_2}}{(1 + \phi)\sqrt{\left[1 + \dfrac{\phi}{(1+\phi)}\right]\lambda_1 + \dfrac{\phi}{(1+\phi)}\left[\lambda_2\left(\lambda_2 + \dfrac{2+\phi}{(1+\phi)}\right)\right]}} \tag{18}
$$

Only in the absence of *non-matching* transplant cases (that is, $\phi = 0$), the primary and repeat transplant cases (namely, the random variables x and y) would have been independent. Otherwise (that is, in the presence of non-matching primary transplant cases), the number of primary and repeat transplant cases are correlated, as their correlation is calculable using expression (18).

Now, we proceed to obtain estimators of the parameters: λ_1, λ_2 and ϕ. For this purpose, consider a bivariate random sample $(x_1, y_1), (x_2, y_2),,,,,,(x_n, y_n)$ from the *seemingly independent bivariate Poisson (SIBP)* distribution (1). From expression (12), notice that the estimator of the rate of repeat transplants is $\hat{\lambda}_2 = \bar{x}$. Using expression (6), the estimator of the rate of primary transplant cases is $\hat{\lambda}_1 = \left(\bar{y} - \dfrac{\phi}{(1+\phi)}\bar{x}\right) \approx \bar{y} - \bar{x}$. Let $r = C\hat{o}rr(x, y)$ denote the sample correlation coefficient. Then, the estimator of the *non-matching level (or the bonding parameter otherwise called)*, ϕ is $\hat{\phi} \approx r\sqrt{\dfrac{\bar{y} - \bar{x}}{\bar{x}}}$.

Illustration of Primary versus Repeat Organ Tranplant Cases in USA

We now examine how the conceptual ideas and the derived expressions of earlier section play out in the 747,594 primary and 91,495 cases of repeat transplant cases across Eastern, Central, Mountain and Pacific regions in USA. The data for analyses and interpretations in this article are downloaded from the public domain http://optn.transplant.hrsa.gov. The primary and repeat transplant cases are categorized in the Table 1-4. The correlation between the primary and repeat transplant cases is calculated and displayed in Fig. 10 for the Eastern, Fig. 11 for the Central, in Fig. 12 for

the Mountain and in Fig. 13 for the Pacific region of the USA. The estimate $\hat{\lambda}_2$ of the rate of repeat transplant cases, $\hat{\lambda}_1$ of the rate of primary transplant cases and $\hat{\phi}$ of the non-matching level of the primary transplant cases (also called the bonding parameter) are calculated and displayed in the tables. The projectile, $\hat{E}[y|x]$ of the number of primary transplant cases for a known number, $X = x$ of repeat transplant cases using (15) and the projectile, $\hat{E}[x|y]$ of the number of repeat transplant cases for a known number, $Y = y$ of primary transplants using (9) are calculated and displayed in the tables. Their risks:

$$\frac{\hat{E}[y|x]}{\hat{V}\{\hat{E}[y|x]\} + \hat{E}\{\hat{V}[y|x]\}}$$

and:

$$\frac{\hat{E}[x|y]}{\hat{V}\{\hat{E}[x|y]\} + \hat{E}\{\hat{V}[x|y]\}}$$

are calculated using (16) and (10) respectively and are separately displayed in the tables for the Eastern, Central, Mountain and Pacific regions in USA. In all regions, the risk for the projectiles of the repeat transplant cases is more in comparison to the risk for the projectiles of the primary transplant cases.

Comments and Conclusion

We summarize that the patterns are consistent across all (Eastern, Central, Mountain and Pacific) regions of the USA. There exists a significant discrepancy between the observed number, $Y = y$ of primary and its projectile, $\hat{E}[y|x]$ as much discrepancy between the observed number, $X = x$ of repeat and its projectile, $\hat{E}[x|y]$. There ought to be reasons for the discrepancies. Perhaps, the discrepancies are attributable not only to the inability of the model "*Seemingly Independent Bivariate Poisson (SIBP) distribution*" to perfectly resemble the actual reality of the transplant occurrences but also to the fact that some of the primary transplant cases might not have enough financial support to go for another round of repeat transplant, died due to other illnesses, or lost from a follow-up in the data collection process. The organ or tissue transplants are not cheap anymore. Some health insurances do reimburse their clients' air travel and/or lodging expenses to overseas hospitals in other countries if the overall medical expense for transplants are lesser expensive. However, this article provides a novel

bivariate distribution and a statistical methodology to analyze and interpret the demand versus supply of organs for transplant within USA.
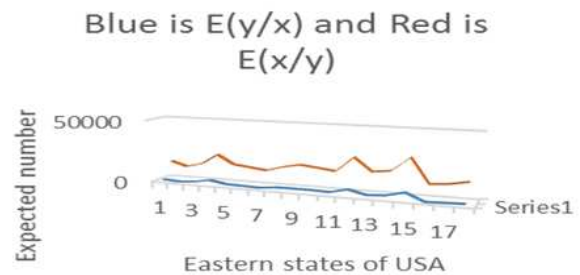

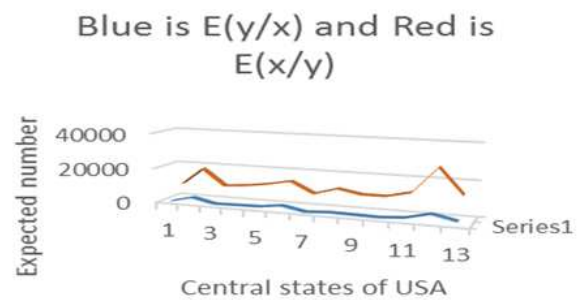Fig. 6. Projectiles for the Eastern states of USA

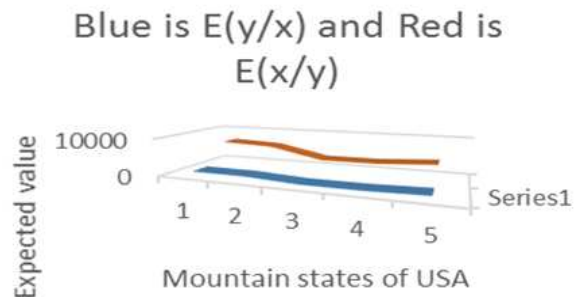Fig. 7. Projectiles for the Central states of USA

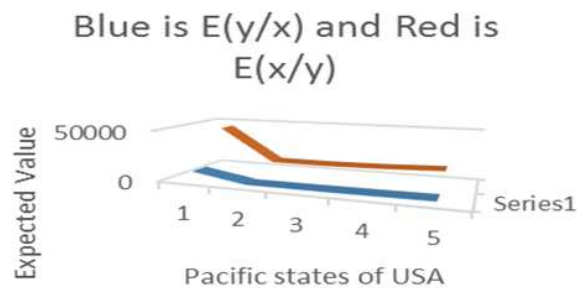Fig. 8. Projectiles for the Mountain states of USA

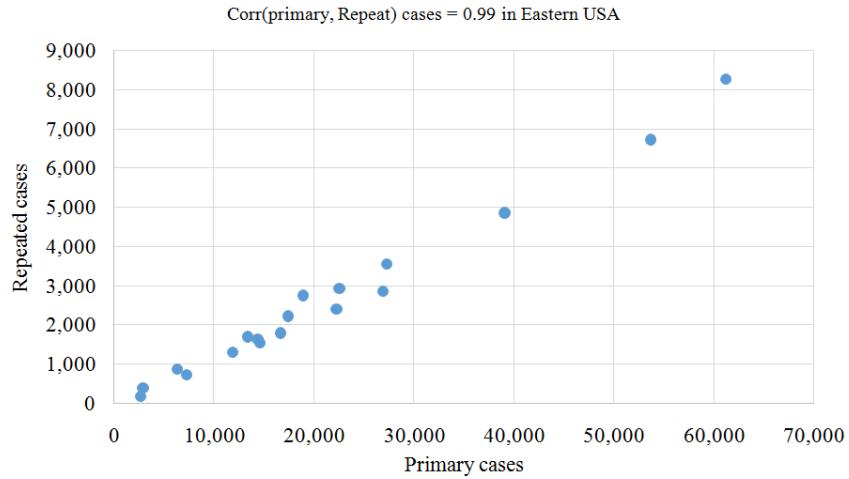Fig. 9. Projectiles for the Pacific states of USA

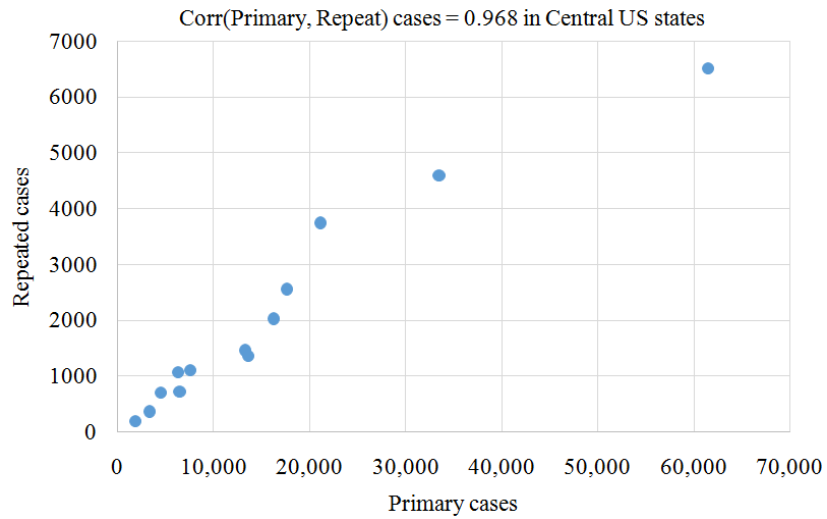Fig. 10. For Eastern US states in 2014

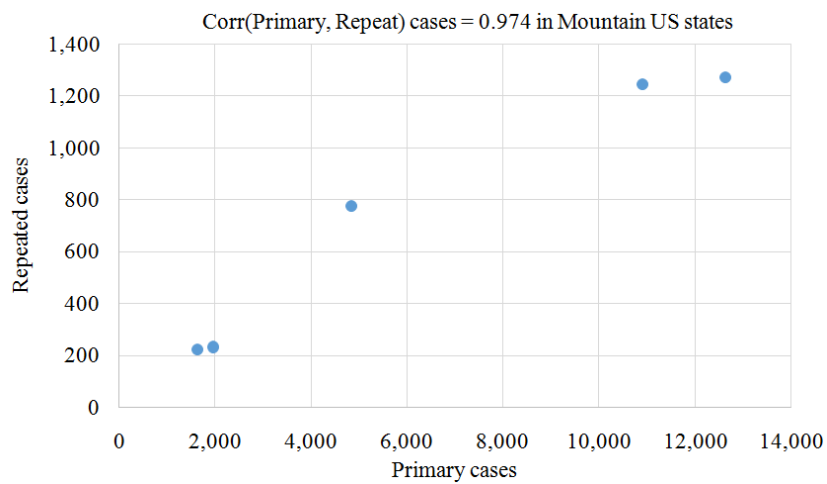Figure 11. For Central US states in 2014
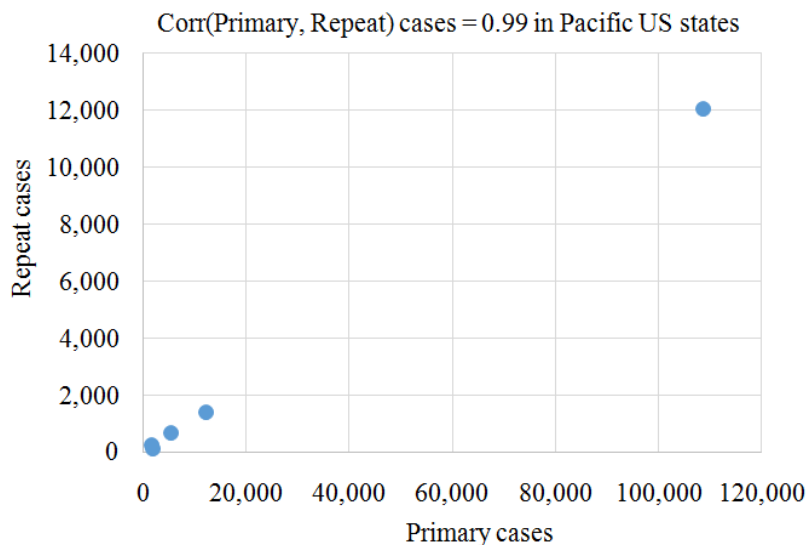

Fig. 12. For Mountain US states in 2014

11

Fig. 13. For Pacific US states in 2014

A comparison (Fig. 6-9) of the conditional projectile, $\hat{E}[y|x]$ for the primary transplant cases versus the projectile, $\hat{E}[x|y]$ for the repeat transplant cases in all (that is, Eastern, Central, Mountain and Pacific) regions of the USA is made. It reveals that the conditional projectile of repeat transplant cases for given number of primary transplant cases is more consistent than the conditional projectile of the primary transplant cases for given number of repeat transplant cases. The consistency is captured using the expressions for risk. The expressions $\dfrac{\hat{E}[y|x]}{\hat{V}\{\hat{E}[y|x]\}+\hat{E}\{\hat{V}[y|x]\}}$ and $\dfrac{\hat{E}[x|y]}{\hat{V}\{\hat{E}[x|y]\}+\hat{E}\{\hat{V}[x|y]\}}$ do respectively estimate the risk of our estimate of primary transplant cases and our estimate of repeat transplant cases. Table 1-4 for the projectiles $\hat{E}[y|x]$ and $\hat{E}[x|y]$. Interestingly, the risk for projecting the number of primary transplant cases based on the number of repeat transplant cases is much smaller than the risk for projecting the number of repeat transplant cases based on the number of primary transplant cases.

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Author's Contributions

The contents of this article are the opinion and thoughts of the author.

Ethics

There is no violation of ethical standards in the contents.

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